Differentiaal Vergelijkingen In de Aardwetenschappen

Series - chapt 1, sections 1-13

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new term, definition

Exercise for werkcollege

Homework

pay attention to this

http://cedricthieulot.net

Arithmetic series

Let us consider the following arithmetic progressions :

$$1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots$$

or

$$0, 5, 10, 15, 20, 25, 30, 35, \ldots$$

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It also follows that

$$a_n = a_1 + (n-1)d$$

The sum S_n of the members of a finite arithmetic progression is called an arithmetic series :

$$S_n = \frac{n}{2}(a_1 + a_n)$$

Geometric series

- Simple examples :

$$2, 4, 8, 16, \ldots$$

$$1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots$$

$$a, ar, ar^2, ar^3, \ldots$$

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$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots \qquad \rightarrow \text{ratio } r = -\frac{1}{2}$$

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- A geometric series has a sum if and only if $|r| < 1$ and in this case

$$S = \frac{a}{1-r}$$

  The series is then called convergent

Ex. 1.1.12, 1.1.13, 1.1.15

Geometric series (3) - Fun stuff

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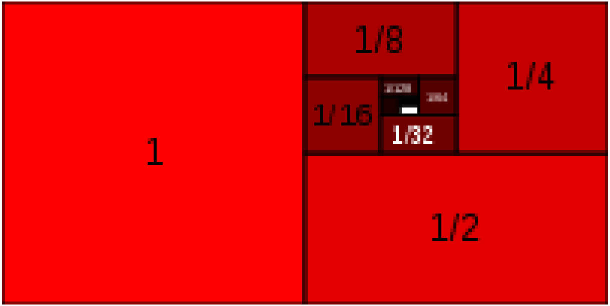

the geometric series 1 + 1/2 + 1/4 + 1/8 + ... converges to 2.

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Harmonic series (1)

the harmonic series is given by

$$\sum_{n=1}^{\infty} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

Question : is the sum finite or infinite ?

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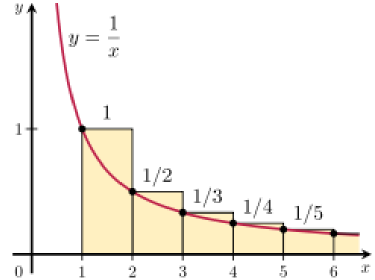

area of rectangles $= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots .$

Harmonic series (2)

$$1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8})+\cdots > 1+\frac{1}{2}+(\frac{1}{4}+\frac{1}{4})+(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8})+\ldots$$

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The harmonic series diverges very slowly : the sum of the first 10^{43} terms is less than 100.

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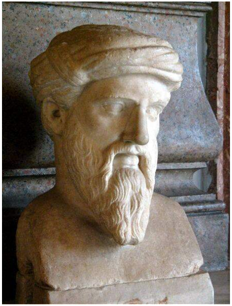

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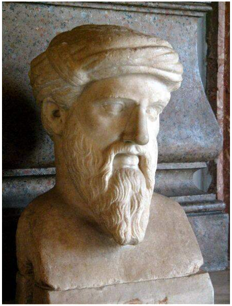

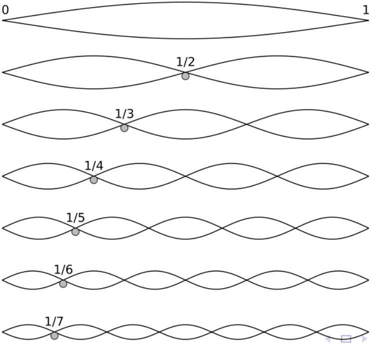

Fibonacci numbers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144$$

Question : is this an arithmetic/geometric/harmonic progression ?

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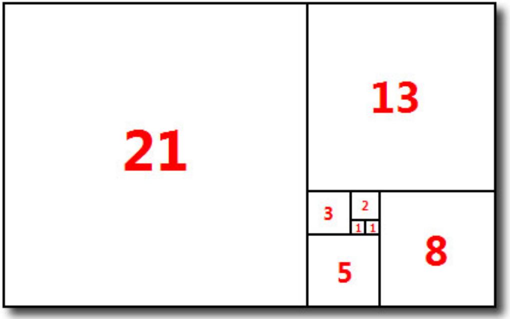

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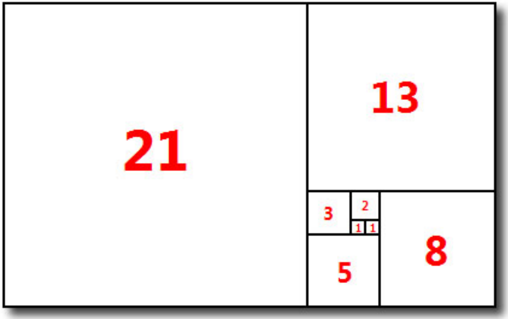
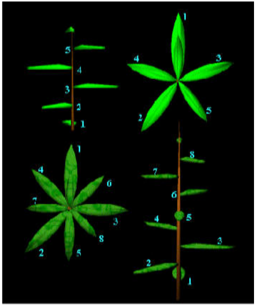
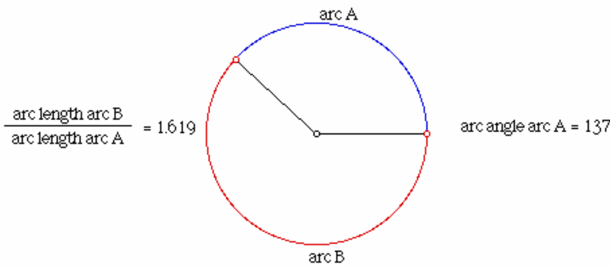


Fibonacci numbers (2)

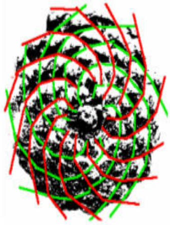
- The Fibonacci numbers are Nature's numbering system
- Plants do not know about this sequence - they just grow in the most efficient way
- Phyllotaxis is the study of the ordered positions of leaves on a stem

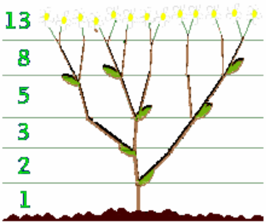




Fibonacci numbers (3)

3 petals	lily, iris
5 petals	buttercup, wild rose, larkspur, columbine
8 petals	delphiniums
13 petals	ragwort, corn marigold, cineraria
21 petals	aster, black-eyed susan, chicory
34 petals	plantain, pytethrum
55,89 petals	michelmas daisies, the asteraceae family





Definitions and notation

- In general an infinite series means an expression of the form

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- $$x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{(n-1)!}$$

$$n! = n \times (n-1) \times (n-2) \times \ldots 1 \qquad \text{with } 0! = 1$$

Ex. 1.2.1

Convergent and divergent series

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- The limiting value S is called the sum of the series.
- The difference $R_n = S - S_n$ is called the remainder.

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} (S - S_n) = S - S = 0$$

Ex. 1.4.3

The preliminary test

Preliminary test :

- if the terms of an infinite series *do not* tend to zero (i.e. $\lim_{n\to\infty} a_n \neq 0$), the series diverges.
- If $\lim_{n\to\infty} a_n = 0$, further testing is needed.

This is *not* a test for convergence.

Ex. 1.5.3

Tests for convergence of series of positive terms (1)

A :The comparison test

- Let

$$m_1 + m_2 + m_3 + m_4 + \dots$$

be a series of positive terms which is convergent.
The series

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is absolutely convergent if $|a_n| \leq m_n$

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- Let
$$d_1 + d_2 + d_3 + d_4 + \ldots$$
be a series of positive terms which is divergent.
The series
$$|a_1| + |a_2| + |a_3| + |a_4| + \ldots$$
diverges if $|a_n| \geq d_n$ for all n from some point on.

Ex. 1.6.13, 1.6.10

Tests for convergence of series of positive terms (2)

B :The integral test

- If $0 < a_{n+1} < a_n$ for $n > N$, then $\sum_{\infty} a_n$ converges if $\int^{\infty} a_n \, dn$ is finite and diverges if the integral is infinite.

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- Example : let us consider the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

Using the integral test :

$$\int^{\infty} \frac{1}{n} dn = [\ln n]^{\infty} = \infty$$

The integral is infinite $\rightarrow$ the series diverges.

Tests for convergence of series of positive terms (3)

C :The ratio test

Let us define ρ_n as follows :

$$\rho_n = \left| \frac{a_{n+1}}{a_n} \right|$$

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Example : $a_n = \frac{1}{n!}$

$$\rho_n = \left| \frac{1/(n+1)!}{1/n!} \right| = \frac{n!}{(n+1)!} = \frac{n(n-1)(n-2)\ldots 1}{(n+1)n(n-1)(n-1)\ldots 1} = \frac{1}{n+1}$$

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$$\rho = \lim_{n\to\infty} \rho_n = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

Since $\rho < 1$ the series converges.

Tests for convergence of series of positive terms (4)

D :Special comparison test

This test has two parts :

- If $\sum_{n=1}^{\infty} b_n$ is a convergent series of positive terms and $a_n \geq 0$ and a_n/b_n tends to a finite limit, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\sum_{n=1}^{\infty} d_n$ is a divergent series of positive terms and $a_n \geq 0$ and a_n/d_n tends to a limit greater than 0 (or tends to $+\infty$), then $\sum_{n=1}^{\infty} a_n$ diverges.

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$$\sum_{n=3}^{\infty} \frac{\sqrt{2n^2 - 5n + 1}}{4n^3 - 7n^2 + 2}$$

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$$2n^2 - 5n + 1 \simeq 2n^2$$

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$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\sqrt{2n^2 - 5n + 1}}{4n^3 - 7n^2 + 2} / \frac{1}{n^2} \right) = \lim_{n\to\infty} \frac{\sqrt{2 - 5/n + 1/n^2}}{4 - 7/n + 2/n^3} = \frac{\sqrt{2}}{4}$$

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Since this is a finite limit, the series a_n converges. Tadaaa !

Alternating series

Example :

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n}$$

Test : An alternating series converges if the absolute value of the terms decreases steadily to zero, that is $|a_{n+1}| \leq |a_n|$ and $\lim_{n\to\infty} a_n = 0$

Since $\frac{1}{n+1} < \frac{1}{n}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$ then the series converges.

Useful facts about series

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  - The resulting series is convergent.
  - its sum is obtained by adding or substracting the sums of the two given series.
- the terms of an absolutely convergent series may be rearranged in any order without affecting either the convergence or sum.

Power series (1)

A power series is of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

or

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Power series (1)

A power series is of the form

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$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!}$$

Power series (2) - interval of convergence

Looking at

$$1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots + \frac{(-x)^n}{2^n}$$

$$\rho_n = \left| \frac{(-x)^{n+1}}{2^{n+1}} \middle/ \frac{(-x)^n}{2^n} \right| = \left| \frac{x}{2} \right|$$

$$\rho = \lim_{n\to\infty} \rho_n = \left| \frac{x}{2} \right|$$

The series converges for $\rho < 1$, i.e. $|x| < 2$.

Mathematical intermezzo

- The concept of a Taylor series was formally introduced by the English mathematician Brook Taylor in 1715.
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Brook Taylor (1685-1731)

Colin Maclaurin (1698-1746)

Power series (3) - theorems

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- two power series may be added, substracted or multiplied ; the resultant series converges at least in the common interval of convergence.
- the power series of a function is unique : there is just one power series of the form $\sum a_n x^n$ which converges to a given function.

Expanding functions in power series

Definition : In mathematics, a Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point.

The Taylor series for $f(x)$ about $x = x_0$ writes :

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2!}(x - x_0)^2 f''(x_0) + \cdots + \frac{1}{n!}(x - x_0)^n f^{(n)}(x_0)$$

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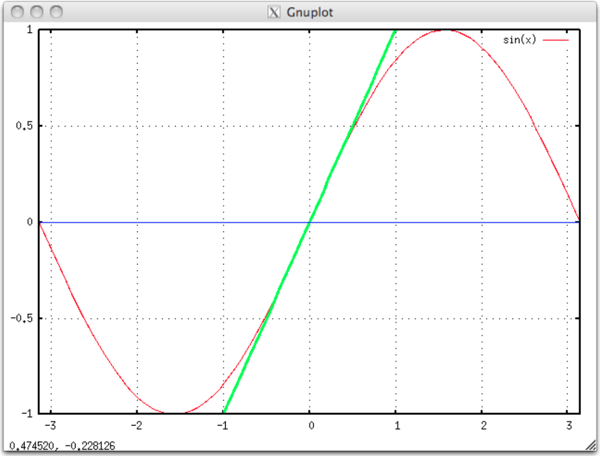
Putting $x_0 = 0$ we obtain the Maclaurin series for $f(x)$:

$$f(x) = f(0) + xf'(0) + \frac{1}{2!}x^2 f''(0) + \cdots + \frac{1}{n!}x^n f^{(n)}(0)$$

Expanding functions in power series (2)

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$



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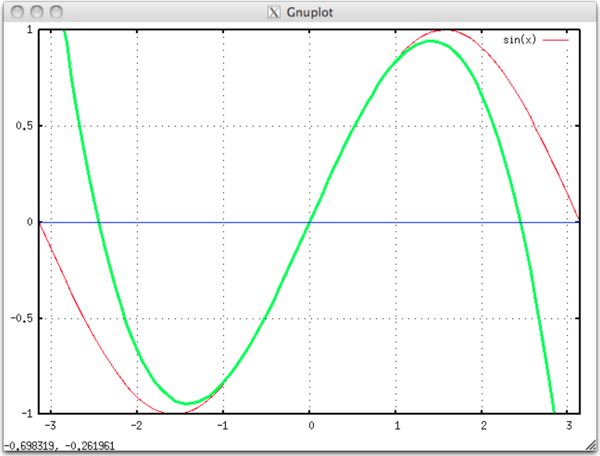

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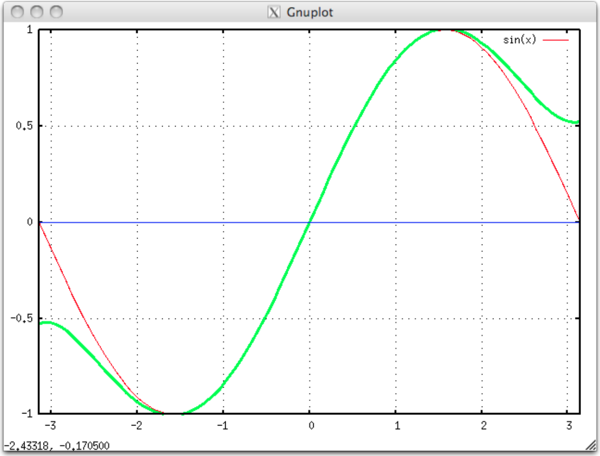

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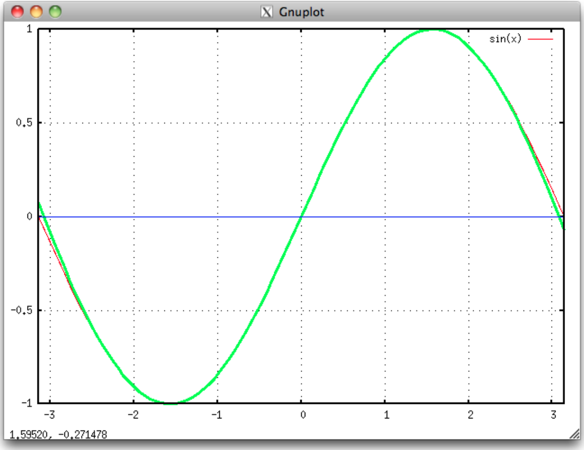

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Question : What are the first terms of the Taylor expansion of a polynomial ?

Let us consider the following 4th-order polynomial expression :

$$f(x) = 4x^4 + 3x^3 - 2x^2 + x - 7$$

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$$\begin{aligned} f'(x) &= 16x^3 + 9x^2 - 4x + 1 &\rightarrow\quad f'(0) &= 1 \\ f''(x) &= 48x^2 + 18x - 4 &\rightarrow\quad f''(0) &= -4 \\ f'''(x) &= 96x + 18 &\rightarrow\quad f'''(0) &= 18 \\ f''''(x) &= 96 &\rightarrow\quad f''''(0) &= 96 \\ f'''''(x) &= 0 &\rightarrow\quad f'''''(0) &= 0 \end{aligned}$$

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$$f(x) = f(0) + xf'(0) + \frac{1}{2!}x^2 f''(0) + \frac{1}{3!}x^3 f'''(0) + \frac{1}{4!}x^4 f''''(0) + 0 + 0 + \dots$$

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$$f(x) = -7 + 1x + \frac{-4}{2}x^2 + \frac{1}{6}x^3 18 + \frac{1}{24}x^4 96 + 0 + 0 + \ldots$$

Tadaa ! the Maclaurin expansion of a polynomial is exactly itself.

Expanding functions in power series (4)

Open a book and you will find

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

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&= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)(2n)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}
\end{aligned}$$

Expanding functions in power series (3)

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \ln(1+x) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \\ \sqrt{1+x} &= \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(1-2n)(n!)^2(4^n)} x^n \quad \text{for} \quad |x| \leq 1 \end{aligned}$$

... and many more !

Computing π

We have

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + ..$$

and we know that

$$\int \frac{1}{1+x^2} = atan(x)$$

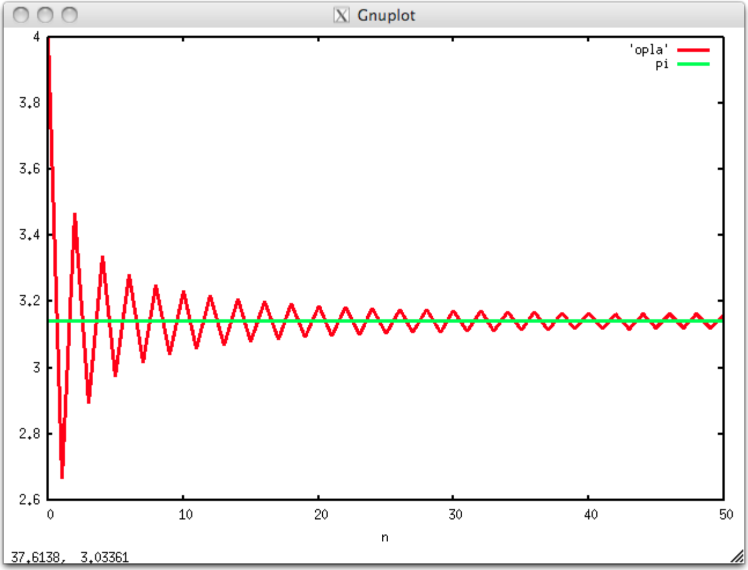
so that

$$atan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

Since $atan(1) = \frac{\pi}{4}$ then

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...\right)$$

Computing π (2)



Application

Suppose we want to evaluate the definite integral

$$\int_0^1 \sin(x^2)dx$$

We know that

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots$$

If we now substitute $t = x^2$ then

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots$$

and then

$$\int_0^1 \sin(x^2)dx = \int_0^1 \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots\right) = \frac{1}{3} - \frac{1}{7.3!} + \frac{1}{11.5!} - \frac{1}{15.7!} + \dots$$