Differentiaal Vergelijkingen In de Aardwetenschappen

Fourier analysis - chapt 7, sections 1-13

C. Thieulot (c.thieulot@uu.nl)

November 2015

new term, definition

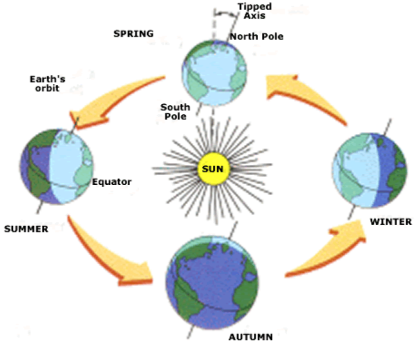
Exercise for werkcolege

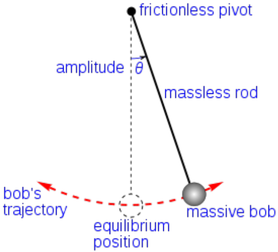
Homework

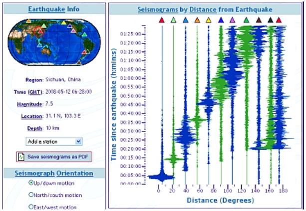
pay attention to this

Introduction

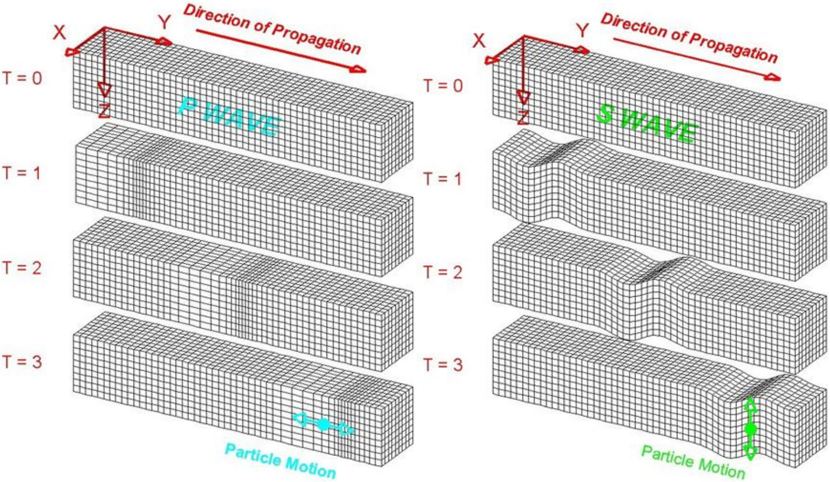
Problems involving vibrations or oscillations occur frequently in physics :

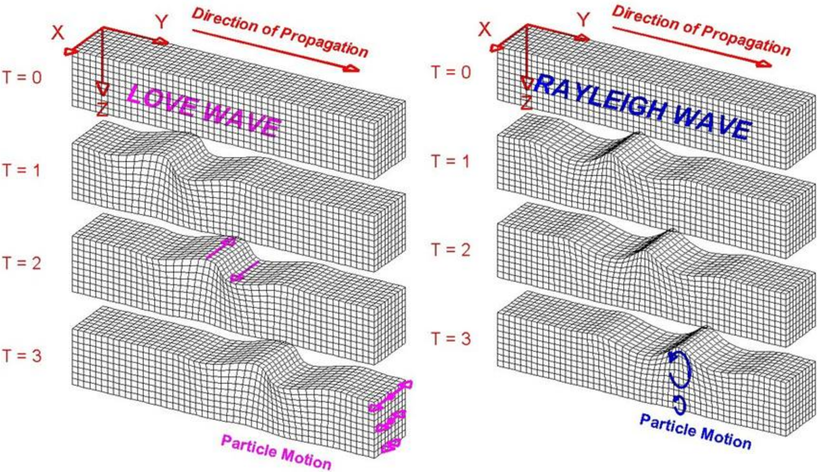






http://en.wikipedia.org/wiki/Kelvin-Helmholtz_instability


X
Y
Z
Direction of Propagation
T = 0
T = 1
T = 2
T = 3
P WAVE
Particle Motion
X
Y
Z
Direction of Propagation
T = 0
T = 1
T = 2
T = 3
S WAVE
Particle Motion



X
Y
Z
Direction of Propagation
T = 0
T = 1
T = 2
T = 3
LOVE WAVE
Particle Motion
RAYLEIGH WAVE

Simple harmonic motion : periodic functions

- particle P moves at constant angular velocity ω around a circle of radius A
- at the same time, let particle Q move up and down along the segment RS such that $y_Q = y_P \ \forall t$

Let $\theta = 0$ at $t = 0$. Then the angle *theta* is given by

$$\theta(t) = \omega \ t$$

Simple harmonic motion : periodic functions

- particle P moves at constant angular velocity ω around a circle of radius A
- at the same time, let particle Q move up and down along the segment RS such that $y_Q = y_P \; \forall t$

Let $\theta = 0$ at $t = 0$. Then the angle *theta* is given by

$$\theta(t) = \omega \, t$$

and the y coordinates of both P and Q :

$$y(t) = A \sin \theta = A \sin(\omega t)$$

Simple harmonic motion : periodic functions

- particle P moves at constant angular velocity ω around a circle of radius A
- at the same time, let particle Q move up and down along the segment RS such that $y_Q = y_P \ \forall t$

Let $\theta = 0$ at $t = 0$. Then the angle *theta* is given by

$$\theta(t) = \omega\ t$$

and the y coordinates of both P and Q :

$$y(t) = A \sin\theta = A \sin(\omega t)$$

The back and forth motion of Q is called simple harmonic motion
If we think of P moving in a complex plane, then

$$z_P = x + iy = A\cos(\omega t) + iA\sin(\omega t) = Ae^{i\omega t}$$

The period is the time for one complete oscillation, that is $2\pi/\omega$

Simple harmonic motion : periodic functions (2)

By definition, the function $f(x)$ is periodic if

$$f(x+P) = f(x)$$

for every x. The number P is the period.

Simple harmonic motion : periodic functions (2)

By definition, the function $f(x)$ is periodic if

$$f(x + P) = f(x)$$

for every x. The number P is the period.

- Example 1 : The period of $f(x) = \sin x$ is 2π.

Simple harmonic motion : periodic functions (2)

By definition, the function $f(x)$ is periodic if

$$f(x+P) = f(x)$$

for every x. The number P is the period.

- Example 1 : The period of $f(x) = \sin x$ is 2π.
- Example 2 : The period of $f(x) = \sin 2\pi x$ is 1, since

$$\sin(2\pi(x+1)) = \sin(2\pi x + 2\pi) = \sin 2\pi x$$

Simple harmonic motion : periodic functions (2)

By definition, the function $f(x)$ is periodic if

$$f(x + P) = f(x)$$

for every x. The number P is the period.

- Example 1 : The period of $f(x) = \sin x$ is 2π.
- Example 2 : The period of $f(x) = \sin 2\pi x$ is 1, since

$$\sin(2\pi(x + 1)) = \sin(2\pi x + 2\pi) = \sin 2\pi x$$

- Example 3 : The period of $f(x) = \cos(\pi x/l)$ is $2l$, since

$$\cos(\pi(x + 2l)/l) = \cos(\pi x/l + 2\pi) = \cos(\pi x/l)$$

Simple harmonic motion : periodic functions (2)

By definition, the function $f(x)$ is periodic if

$$f(x+P) = f(x)$$

for every x. The number P is the period.

- Example 1 : The period of $f(x) = \sin x$ is 2π.
- Example 2 : The period of $f(x) = \sin 2\pi x$ is 1, since

$$\sin(2\pi(x+1)) = \sin(2\pi x + 2\pi) = \sin 2\pi x$$

- Example 3 : The period of $f(x) = \cos(\pi x/l)$ is $2l$, since

$$\cos(\pi(x+2l)/l) = \cos(\pi x/l + 2\pi) = \cos(\pi x/l)$$

In general, the period of $\sin \frac{2\pi x}{T}$ is T.

Simple harmonic motion : periodic functions (3)

The (vertical) motion of Q is given by

$$y(t) = A\sin(\omega t)$$

Its velocity is simply

$$v(t) = \frac{d}{dt}y(t) = A\omega\cos(\omega t)$$

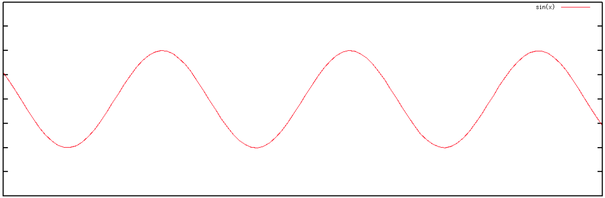
Simple harmonic motion : periodic functions (3)

The (vertical) motion of Q is given by

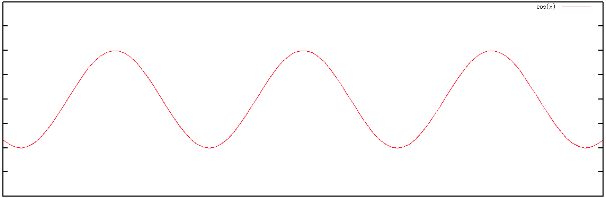
$$y(t) = A\sin(\omega t)$$

Its velocity is simply

$$v(t) = \frac{d}{dt}y(t) = A\omega\cos(\omega t)$$



(position)



(velocity)

Applications of Fourier series (1)

Jean Baptiste Joseph Fourier (1768-1830)
French mathematician and physicist best known for initiating the investigation of Fourier series and their applications to problems of heat transfer and vibrations.
The Fourier transform and Fourier's law are also named in his honour. Fourier is also generally credited with the discovery of the greenhouse effect.

Applications of Fourier series (2) - Soundwaves

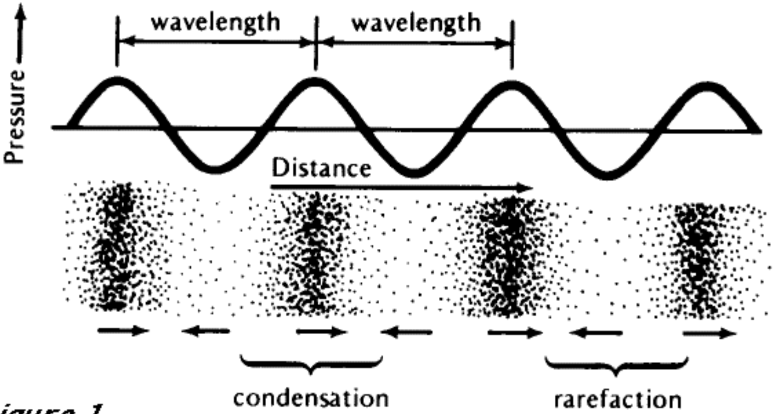
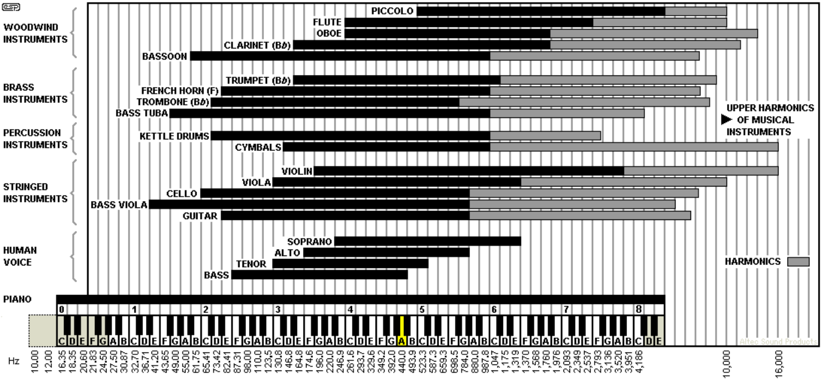


Figure 1

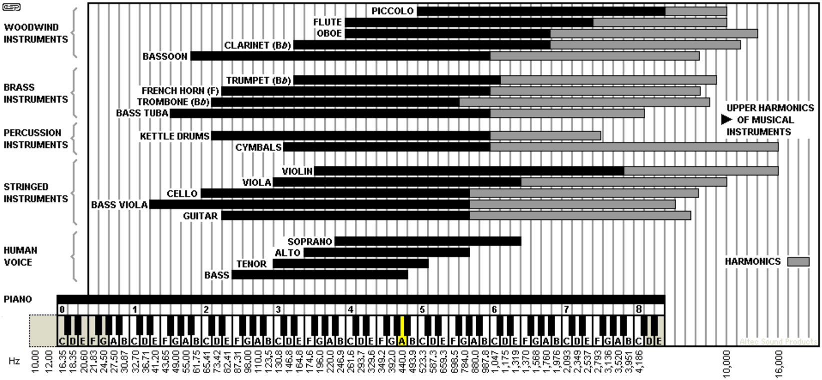
In physics, sound is a vibration that propagates as a typically audible mechanical wave of pressure and displacement, through a medium such as air or water.

sounds frequency range

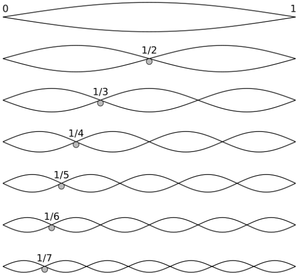


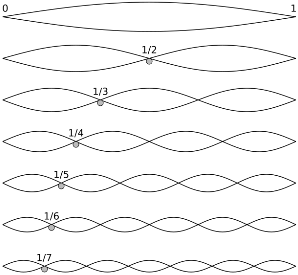
- Human range : 20Hz - 20,000Hz

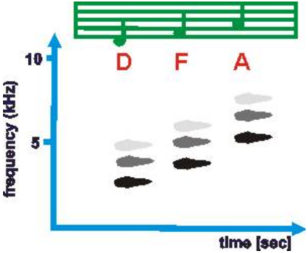
sounds frequency range

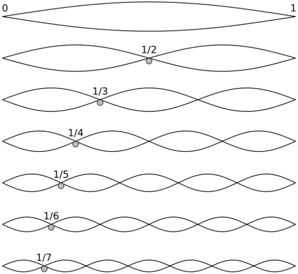


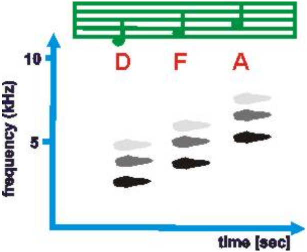
- Human range : 20Hz - 20,000Hz
- cats : 45Hz - 64,000Hz
- cows : 23Hz - 35,000Hz

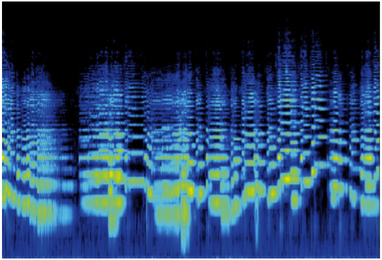

0
1
1/2
1/3
1/4
1/5
1/6
1/7


0
1
1/2
1/3
1/4
1/5
1/6
1/7


D
F
A
10
5
frequency (kHz)
time [sec]

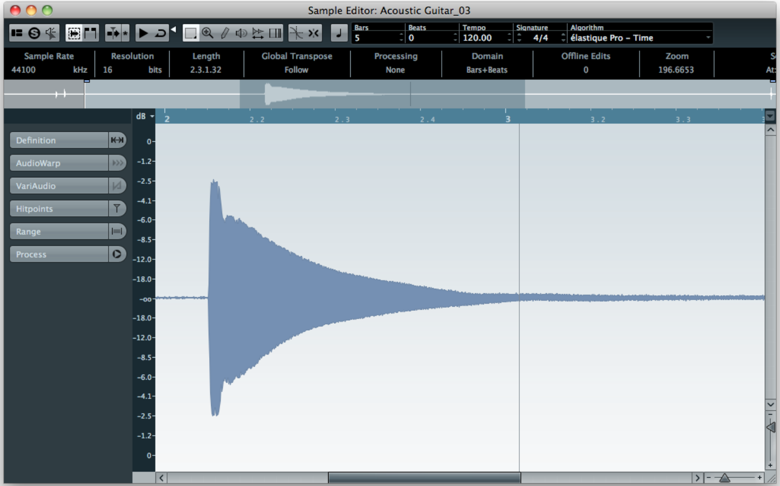




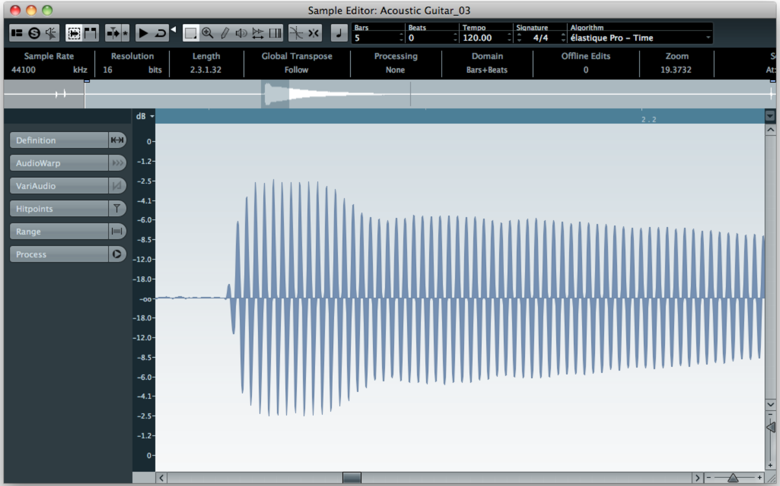


$\rightarrow$ real-life signals are a complicated combination of periodic functions.

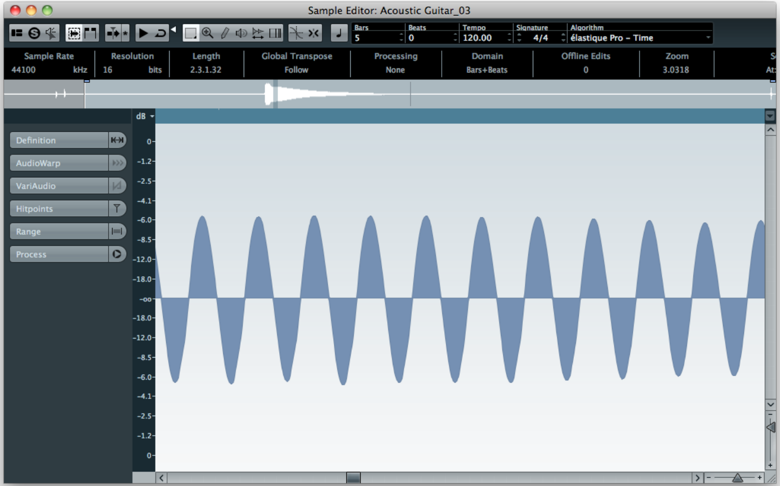
single guitar string waveform (1)



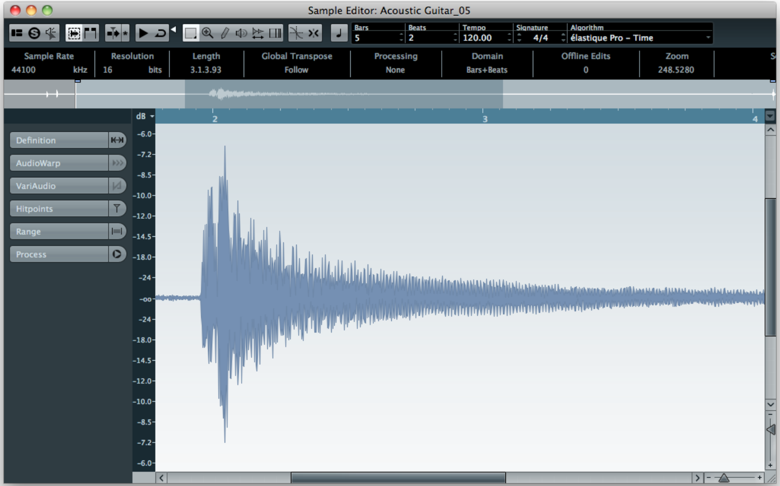
single guitar string waveform (2)



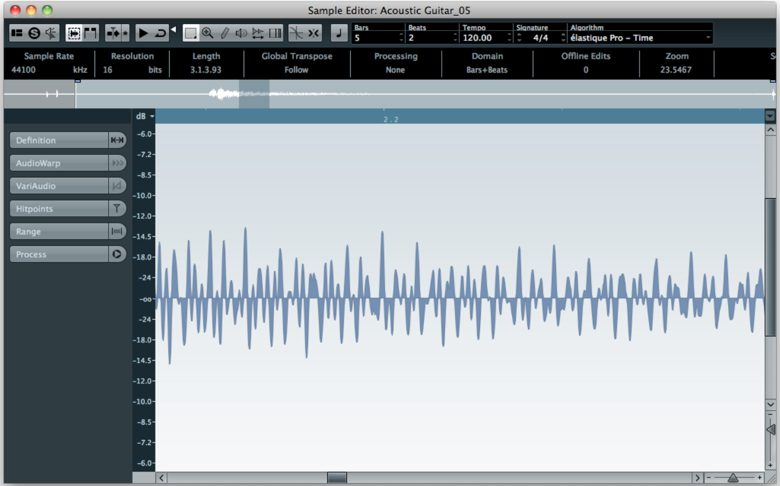
single guitar string waveform (3)



guitar chord waveform (1)

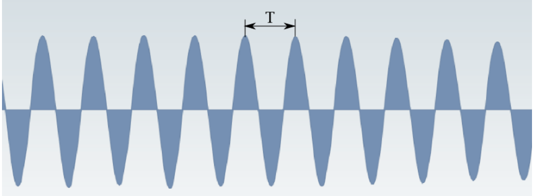


guitar chord waveform (2)



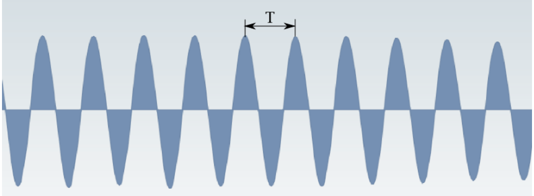
guitar chord waveform (2)

- simple oscillator/source

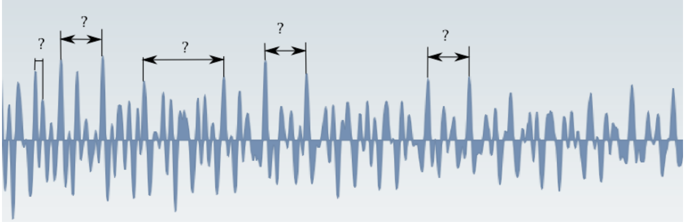


guitar chord waveform (2)

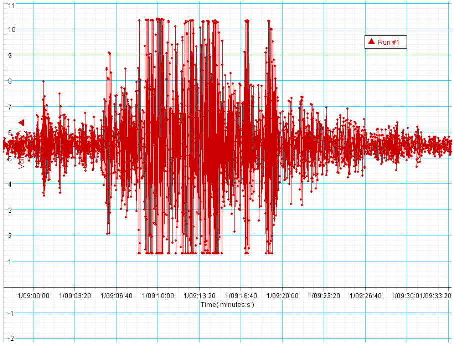
- simple oscillator/source

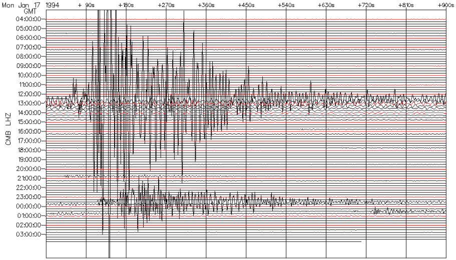


- complex source

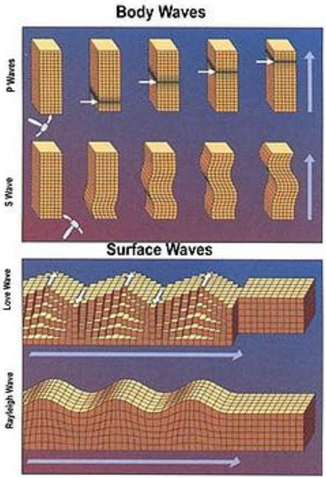


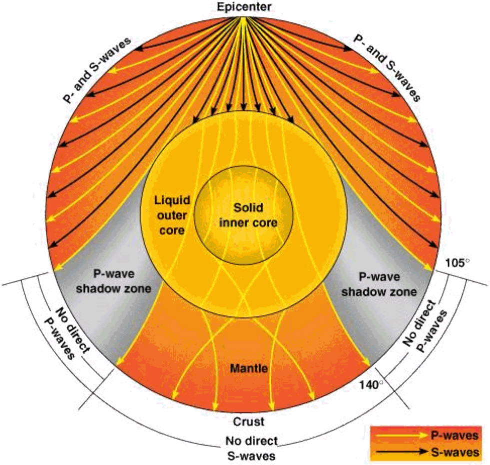
seismograms





seismic waves





→ Complex source + complex medium !

Applications of Fourier series (2)

Question : Given a complicated signal, how can we write it as a sum of terms (fundamental freq. + harmonics) ?

Applications of Fourier series (2)

Question : Given a complicated signal, how can we write it as a sum of terms (fundamental freq. + harmonics) ?

Expanding a function in a Fourier series then amounts to breaking it down into its various harmonics

Applications of Fourier series (2)

Question : Given a complicated signal, how can we write it as a sum of terms (fundamental freq. + harmonics) ?

Expanding a function in a Fourier series then amounts to breaking it down into its various harmonics

$$f(t) = a_1 \sin(\omega_1 t) + a_2 \sin(\omega_2 t) + \dots$$

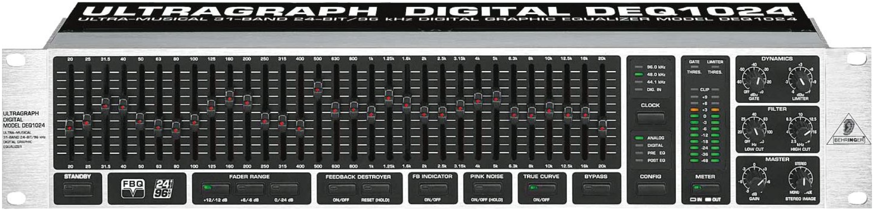
Applications of Fourier series (2)

Question : Given a complicated signal, how can we write it as a sum of terms (fundamental freq. + harmonics) ?

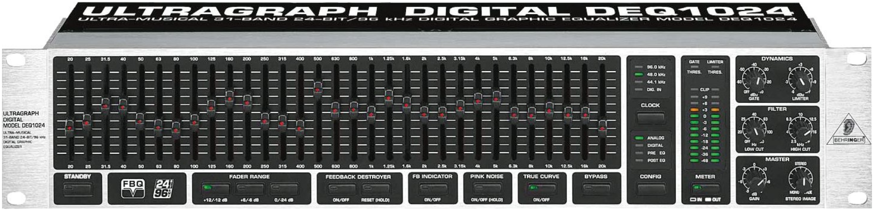
Expanding a function in a Fourier series then amounts to breaking it down into its various harmonics

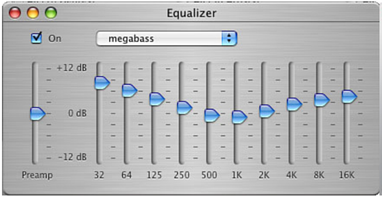
$$f(t) = a_1 \sin(\omega_1 t) + a_2 \sin(\omega_2 t) + \dots$$

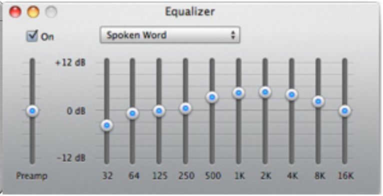
Given $f(t)$, how do we compute $a_1, a_2, \ldots$ and $\omega_1, \omega_2, \ldots$?



ULTRAGRAPH DIGITAL DEQ1024
ULTRA-MUSICAL 31-BAND 24-BIT/96 kHz DIGITAL GRAPHIC EQUALIZER MODEL DEQ1024
ULTRAGRAPH
DIGITAL
MODEL DEQ1024
STANDBY
FBQ
FADER RANGE
+12/-12 dB
+6/-6 dB
0/-24 dB
FEEDBACK DESTROYER
ON/OFF
RESET (HOLD)
FB INDICATOR
ON/OFF
PINK NOISE
ON/OFF (HOLD)
TRUE CURVE
ON/OFF
BYPASS
CLOCK
CONFIG
METER
DYNAMICS
GATE
LIMITER
FILTER
LOW CUT
HIGH CUT
MASTER
GAIN
STEREO IMAGE
BEHRINGER


ULTRAGRAPH DIGITAL DEQ1024
ULTRA-MUSICAL 31-BAND 24-BIT/96 kHz DIGITAL GRAPHIC EQUALIZER MODEL DEQ1024
ULTRAGRAPH DIGITAL MODEL DEQ1024
STANDBY
FBQ
FADER RANGE
+12/-12 dB
+6/-6 dB
0/-24 dB
FEEDBACK DESTROYER
ON/OFF
RESET (HOLD)
FB INDICATOR
ON/OFF
PINK NOISE
ON/OFF (HOLD)
TRUE CURVE
ON/OFF
BYPASS
CLOCK
CONFIG
METER
DYNAMICS
GATE
LIMITER
FILTER
LOW CUT
HIGH CUT
MASTER
GAIN
STEREO IMAGE
BEHRINGER


Equalizer
On
megabass
+12 dB
0 dB
-12 dB
Preamp
32
64
125
250
500
1K
2K
4K
8K
16K


Equalizer
On
Spoken Word
+12 dB
0 dB
-12 dB
Preamp
32
64
125
250
500
1K
2K
4K
8K
16K

Average value of a function

- Let us look at these numbers : 1,5,11,4,7.
  What is their (arithmetic) average ?

Average value of a function

- Let us look at these numbers : 1,5,11,4,7.
  What is their (arithmetic) average ?

$$avrg = \frac{1+5+11+4+7}{5} = 5.6$$

Average value of a function

- Let us look at these numbers : 1,5,11,4,7.
  What is their (arithmetic) average ?

$$avrg = \frac{1+5+11+4+7}{5} = 5.6$$

- The numbers are now obtained through a function f : $f(x_1)$, $f(x_2)$, $f(x_3)$, $f(x_4)$, $f(x_5)$. Their average is still

$$avrg = \frac{f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)}{5}$$

Average value of a function

- Let us look at these numbers : 1,5,11,4,7.
  What is their (arithmetic) average ?

$$avrg = \frac{1+5+11+4+7}{5} = 5.6$$

- The numbers are now obtained through a function f : $f(x_1)$, $f(x_2)$, $f(x_3)$, $f(x_4)$, $f(x_5)$. Their average is still

$$avrg = \frac{f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)}{5}$$

- if there are more than 5 points :

$$avrg = \frac{f(x_1)+f(x_2)+f(x_3)+\cdots+f(x_n)}{n}$$

Average value of a function (2)

- Let us have

  $$a = x_1 \leq x_2 \leq x_3 \cdots \leq x_{n-1} \leq x_n = b$$

  The average of the function f over the interval $[a, b]$ is given by

  $$avrg = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} = \frac{(f(x_1) + f(x_2) + \cdots + f(x_n))\Delta x}{n\Delta x}$$

- the average becomes more precise with $n \to \infty$, i.e. $\Delta x \to 0$. In this case

  $$avrg = \frac{1}{b-a}\int_a^b f(x)dx$$

Maths, hairstyle and history (continued)

- The symbol $\int$ is used to denote the integral in mathematics.

Maths, hairstyle and history (continued)

- The symbol $\int$ is used to denote the integral in mathematics.
- The notation was introduced by the German mathematician Gottfried Wilhelm von Leibniz towards the end of the 17th century.

Maths, hairstyle and history (continued)

- The symbol $\int$ is used to denote the integral in mathematics.
- The notation was introduced by the German mathematician Gottfried Wilhelm von Leibniz towards the end of the 17th century.
- The symbol was based on the "long s" character, and was chosen because Leibniz thought of the integral as an infinite sum of infinitesimal summands.

Maths, hairstyle and history (continued)

- The symbol $\int$ is used to denote the integral in mathematics.
- The notation was introduced by the German mathematician Gottfried Wilhelm von Leibniz towards the end of the 17th century.
- The symbol was based on the "long s" character, and was chosen because Leibniz thought of the integral as an infinite sum of infinitesimal summands.

Average value of a function (4)

Example : The average of $\sin x$ over any integer number of periods is zero :

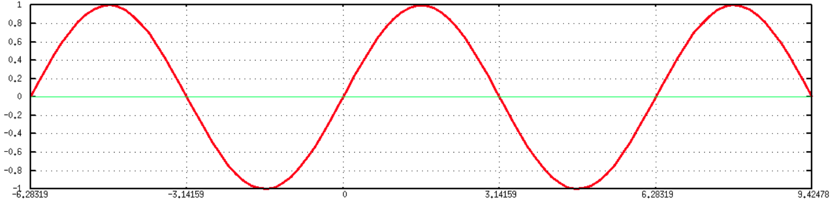
$$\int_0^{2\pi} \sin x \, dx = -[\cos x]_0^{2\pi} = -[\cos 2\pi - \cos 0] = 0$$

Average value of a function (4)

Example : The average of $\sin x$ over any integer number of periods is zero :

$$\int_0^{2\pi} \sin x \; dx = -[\cos x]_0^{2\pi} = -[\cos 2\pi - \cos 0] = 0$$

$$\int_{-\pi}^{\pi} \sin x \; dx = -[\cos x]_{-\pi}^{\pi} = -[\cos(\pi) - \cos(-\pi)] = -((-1) - (-1)) = 0$$



Average value of a function (4)

How can we compute this ?

$$\frac{1}{\pi}\int_0^{\pi} \cos^2 x \, dx$$

Average value of a function (4)

How can we compute this ?

$$\frac{1}{\pi}\int_0^{\pi} \cos^2 x \, dx$$

Let us remember :

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

so that

$$\cos^2 x = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 = \frac{e^{2ix} + 2e^{ix}e^{-ix} + e^{-2ix}}{4} = \frac{e^{2ix} + 2 + e^{-2ix}}{4} = \frac{2\cos 2x + 2}{4}$$

Average value of a function (4)

How can we compute this ?

$$\frac{1}{\pi}\int_0^{\pi} \cos^2 x \; dx$$

Let us remember :

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

so that

$$\cos^2 x = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 = \frac{e^{2ix} + 2e^{ix}e^{-ix} + e^{-2ix}}{4} = \frac{e^{2ix} + 2 + e^{-2ix}}{4} = \frac{2\cos 2x + 2}{4}$$

leading to

$$\int_0^{\pi} \cos^2 x \; dx = \int_0^{\pi} \frac{\cos 2x + 1}{2} dx = \int_0^{\pi} \frac{1}{2}(\cos(2x) + 1) dx$$

Average value of a function (4)

How can we compute this ?

$$\frac{1}{\pi}\int_0^{\pi} \cos^2 x \, dx$$

Let us remember :

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

so that

$$\cos^2 x = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 = \frac{e^{2ix} + 2e^{ix}e^{-ix} + e^{-2ix}}{4} = \frac{e^{2ix} + 2 + e^{-2ix}}{4} = \frac{2\cos 2x + 2}{4}$$

leading to

$$\int_0^{\pi} \cos^2 x \, dx = \int_0^{\pi} \frac{\cos 2x + 1}{2} dx = \int_0^{\pi} \frac{1}{2}(\cos(2x) + 1) dx$$

The period of $\cos 2x$ is π so $\int_0^{\pi} \cos 2x \, dx = 0$ and finally

$$\frac{1}{\pi}\int_0^{\pi} \cos^2 x \, dx = \frac{1}{\pi}\int_0^{\pi} \frac{1}{2} \, dx = \frac{1}{\pi}\frac{\pi}{2} = \frac{1}{2}$$

Fourier coefficients

Let us assume that a function $f(x)$ of period 2π can be decomposed as follows

$$\begin{aligned} f(x) = \frac{1}{2}a_0 \quad &+ \quad a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots \\ &+ \quad b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots \end{aligned}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \ dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \ dx$$

Fourier coefficients (2)

One can prove that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2 nx\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^2 nx\,dx = \frac{1}{2}$$

Fourier coefficients (2)

One can prove that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2 nx\ dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^2 nx\ dx = \frac{1}{2}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin mx\ \cos nx\ dx = 0$$

Fourier coefficients (2)

One can prove that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 nx\; dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2 nx\; dx = \frac{1}{2}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin mx\;\; \cos nx\; dx = 0$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin mx\;\; \sin nx\; dx = \begin{cases} 0 & m \neq n \\ 1/2 & m = n \neq 0 \\ 0 & m = n = 0 \end{cases}$$

Fourier coefficients (2)

One can prove that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2 nx\; dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^2 nx\; dx = \frac{1}{2}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin mx\;\cos nx\; dx = 0$$

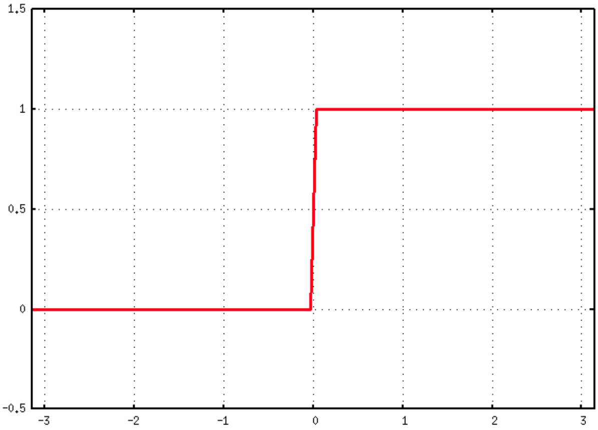
$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin mx\;\sin nx\; dx = \begin{cases} 0 & m \neq n \\ 1/2 & m = n \neq 0 \\ 0 & m = n = 0 \end{cases}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos mx\;\cos nx\; dx = \begin{cases} 0 & m \neq n \\ 1/2 & m = n \neq 0 \\ 1 & m = n = 0 \end{cases}$$

Fourier coefficients - Example 1

Let

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$



Fourier coefficients - Example 1

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \; dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\cos nx \; dx + \frac{1}{\pi}\int_{0}^{\pi} 1\cdot\cos nx \; dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \cos nx \; dx \\
&= \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}
\end{aligned}
$$

Fourier coefficients - Example 1

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \; dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\cos nx \; dx + \frac{1}{\pi}\int_{0}^{\pi} 1\cdot\cos nx \; dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \cos nx \; dx \\
&= \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases} \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \; dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin nx \; dx \\
&= \begin{cases} 0 & \text{even } n \\ 2/n\pi & \text{odd } n \end{cases}
\end{aligned}
$$

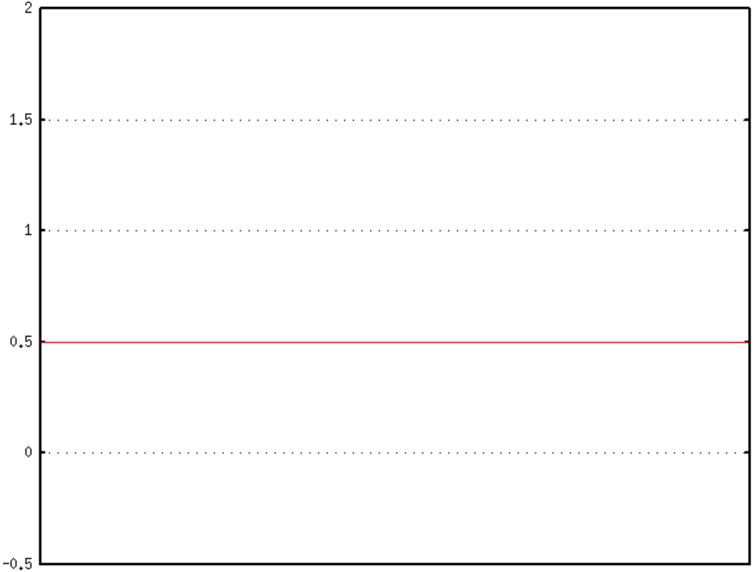
Fourier coefficients - Example 1

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\cos nx\ dx + \frac{1}{\pi}\int_{0}^{\pi} 1\cdot\cos nx\ dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \cos nx\ dx \\
&= \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases} \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin nx\ dx \\
&= \begin{cases} 0 & \text{even } n \\ 2/n\pi & \text{odd } n \end{cases}
\end{aligned}$$

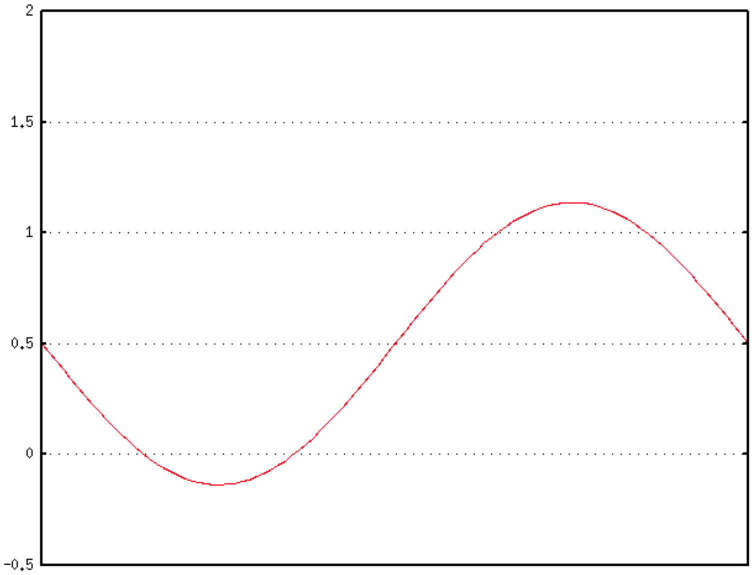
Then

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right)$$

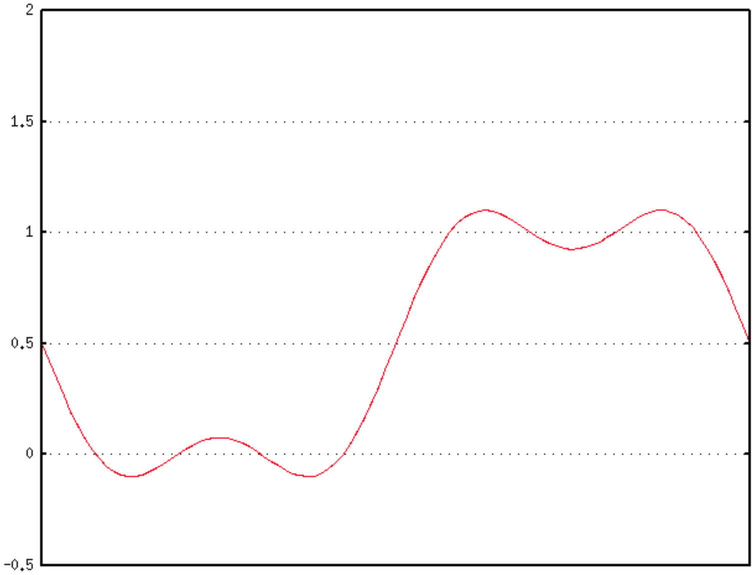
$$f(x) = \frac{1}{2}$$



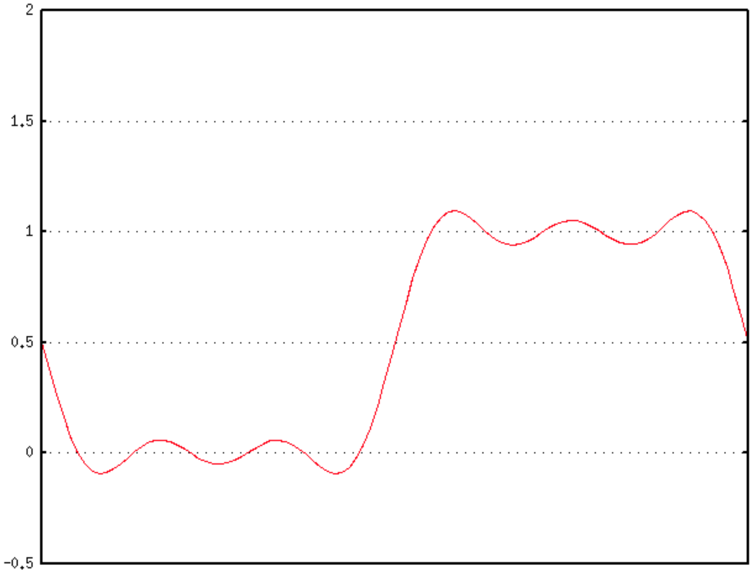
$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\sin x}{1}\right)$$



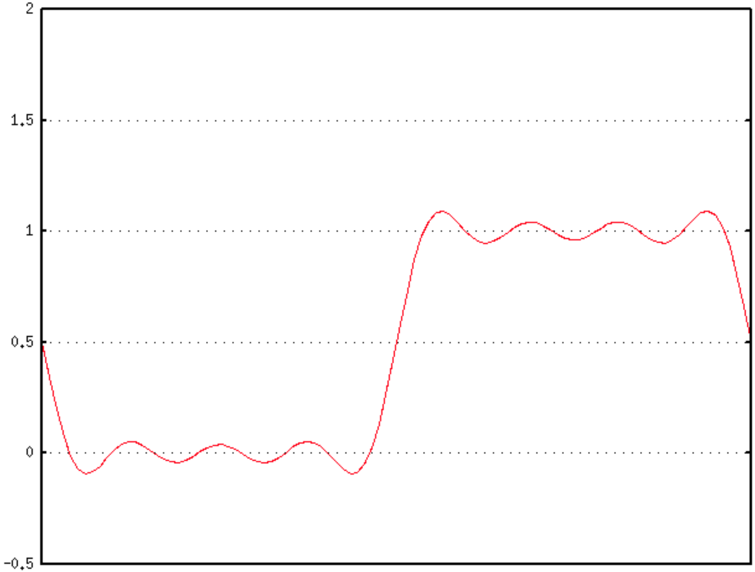
$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3}\right)$$



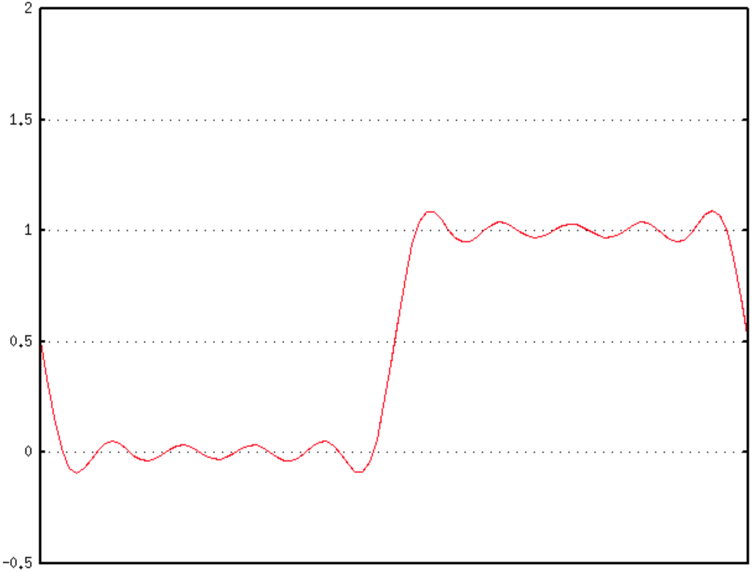
$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5}\right)$$



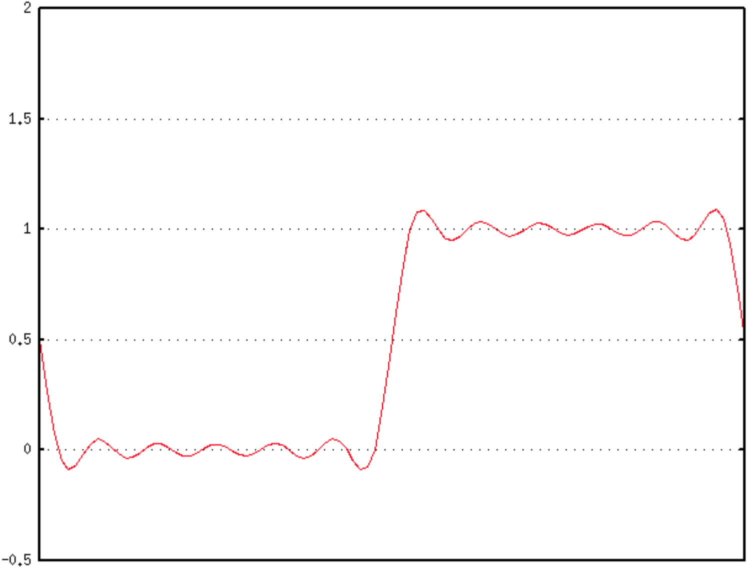
$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7}\right)$$



$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \frac{\sin 9x}{9}\right)$$



$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \frac{\sin 9x}{9} + \frac{\sin 11x}{11}\right)$$



Dirichlet conditions

Question : now that we have a series, does it converge to the values of $f(x)$?

Theorem of Dirichlet : If $f(x)$ is periodic of period 2π, and if between $-\pi$ and $+\pi$ it is single-valued, has a finite number of minimum and maximum values, and a finite number of discontinuities, and if $\int_{-\pi}^{\pi} |f(x)| dx$ is finite, then the Fourier series converge to $f(x)$ at all the points where $f(x)$ is continuous ; at jumps the Fourier series converges to the midpoint of the jump.

Maths, hairstyle and history (continued)

Johann Peter Gustav Lejeune Dirichlet (1805-1859), German mathematician

Dirichlet conditions (2)

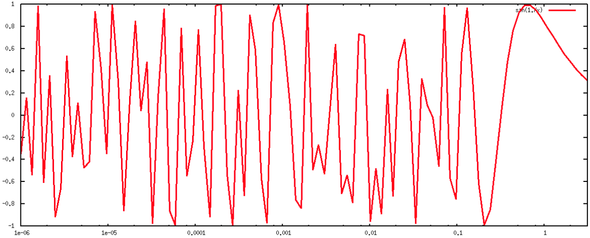
all assumptions in the theorem matter!

- consider $x^2 + y^2 = 1$. In this case y is not a single-valued.

Dirichlet conditions (2)

all assumptions in the theorem matter !

- consider $x^2 + y^2 = 1$. In this case y is not a single-valued.
- consider $f(x) = \sin(1/x)$

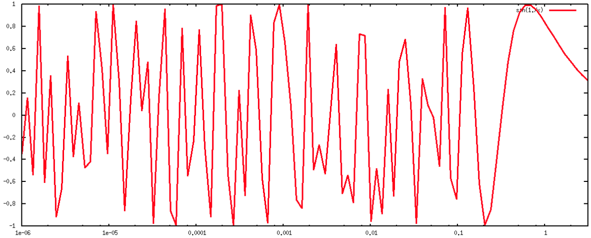


It has an infinit number of minima and maxima between $-\pi$ and π

Dirichlet conditions (2)

all assumptions in the theorem matter !

- consider $x^2 + y^2 = 1$. In this case y is not a single-valued.
- consider $f(x) = \sin(1/x)$



It has an infinit number of minima and maxima between $-\pi$ and π

- consider $f(x) = 1/x$

$$\int_{-\pi}^{+\pi} |f(x)| dx = 2 \int_{0}^{+\pi} \frac{1}{x} dx = 2[\ln x]_0^{\pi} = \infty$$

Complex form of Fourier series

Recall

$$\sin(nx) = \frac{e^{inx} - e^{-inx}}{2i} \qquad \cos(nx) = \frac{e^{inx} + e^{-inx}}{2}$$

We could insert this into

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\; dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\; dx$$

and get a series of terms of the forms e^{inx}, e^{-inx}.

Complex form of Fourier series

Recall

$$\sin(nx) = \frac{e^{inx} - e^{-inx}}{2i} \qquad \cos(nx) = \frac{e^{inx} + e^{-inx}}{2}$$

We could insert this into

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \; dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \; dx$$

and get a series of terms of the forms e^{inx}, e^{-inx}.

This is the complex form of the Fourier series.

Complex form of Fourier series

What if we want to find the coefficients in the complex form directly ? We assume

$$f(x) = c_0 + c_1 e^{ix} + c_{-1} e^{-ix} + c_2 e^{2ix} + c_{-2} e^{-2ix} + \cdots = \sum_{n=-\infty}^{+\infty} c_n e^{inx}$$

Complex form of Fourier series

What if we want to find the coefficients in the complex form directly ? We assume

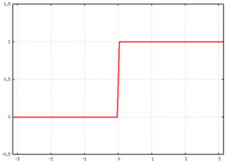
$$f(x) = c_0 + c_1 e^{ix} + c_{-1} e^{-ix} + c_2 e^{2ix} + c_{-2} e^{-2ix} + \cdots = \sum_{n=-\infty}^{+\infty} c_n e^{inx}$$

This leads to

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx} dx$$

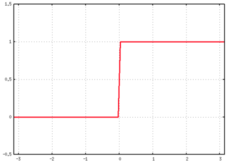
Complex form of Fourier series - Example 1

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$



Complex form of Fourier series - Example 1

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$



$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx} dx = \frac{1}{2\pi}\int_{0}^{\pi} e^{-inx} dx \\ &= \frac{1}{-2\pi in}(e^{-in\pi} - 1) \\ &= \begin{cases} 0 & \text{even } n, n \neq 0 \\ 1/in\pi & \text{odd } n \end{cases} \\ c_0 &= 1/2 \end{aligned}$$

Other intervals

Let us now consider the function $f(x)$ periodic over the interval $[-l, l]$. We now have

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + a_1 \cos\frac{\pi x}{l} + a_2 \cos\frac{2\pi x}{l} + \dots \\
&\quad + b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + \dots \\
&= \frac{a_0}{2} + \sum_{l=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l} \right) \\
f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}
\end{aligned}
$$

Other intervals

Let us now consider the function $f(x)$ periodic over the interval $[-l, l]$. We now have

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + a_1 \cos\frac{\pi x}{l} + a_2 \cos\frac{2\pi x}{l} + \ldots \\
&\quad + b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + \ldots \\
&= \frac{a_0}{2} + \sum_{l=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \\
f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}
\end{aligned}
$$

with

$$
\begin{aligned}
a_n &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\, dx \\
b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\, dx \\
c_n &= \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-in\pi x/l} dx \qquad (1)
\end{aligned}
$$

Other intervals

Example :

$$f(x) = \begin{cases} 0 & 0 < x < l \\ 1 & l < x < 2l \end{cases}$$

$$\begin{aligned} c_n &= \frac{1}{2l}\int_0^l 0 \cdot dx + \frac{1}{2l}\int_l^{2l} 1 \cdot e^{-i\pi nx/l} dx \\ &= \frac{1}{2l}\left[\frac{e^{-in\pi x/l}}{-i\pi n/l}\right]_l^{2l} \\ &= \frac{1}{-2i\pi n}(1 - e^{in\pi}) \\ &= \begin{cases} 0 & \text{even } n, n \neq 0 \\ -1/in\pi & \text{odd } n \end{cases} \\ c_0 &= \frac{1}{2} \end{aligned}$$

Even and odd functions

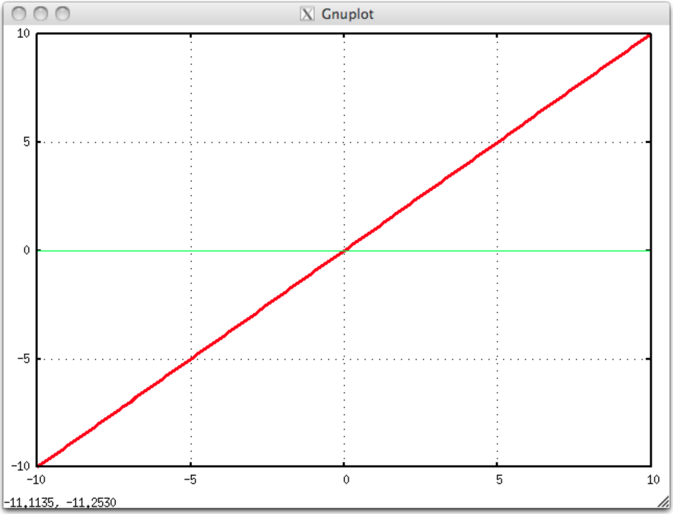
- the function $f(x)$ is even if

$$f(-x) = f(x)$$

- the function $f(x)$ is odd if

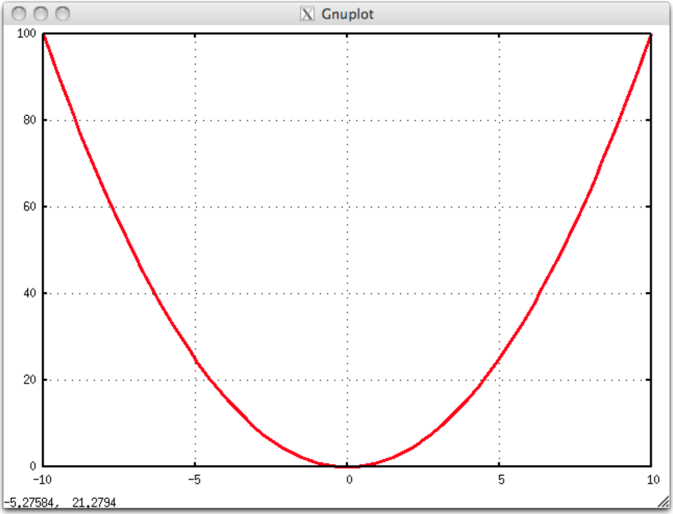
$$f(-x) = -f(x)$$

Even and odd functions (2)



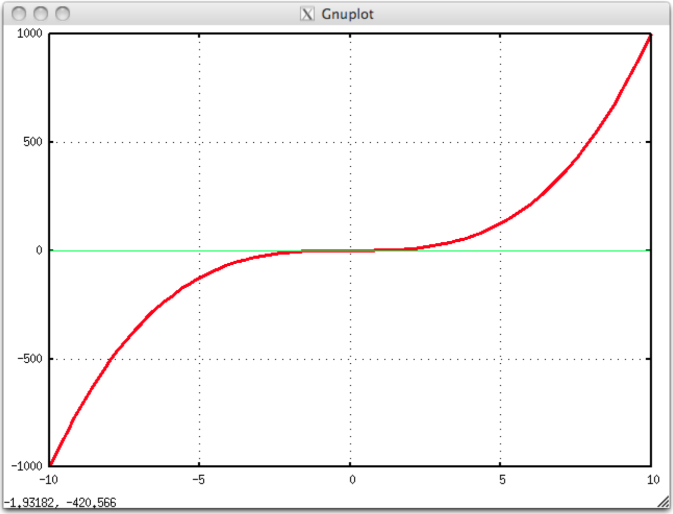
$f(x) = x$ is odd.

Even and odd functions (3)



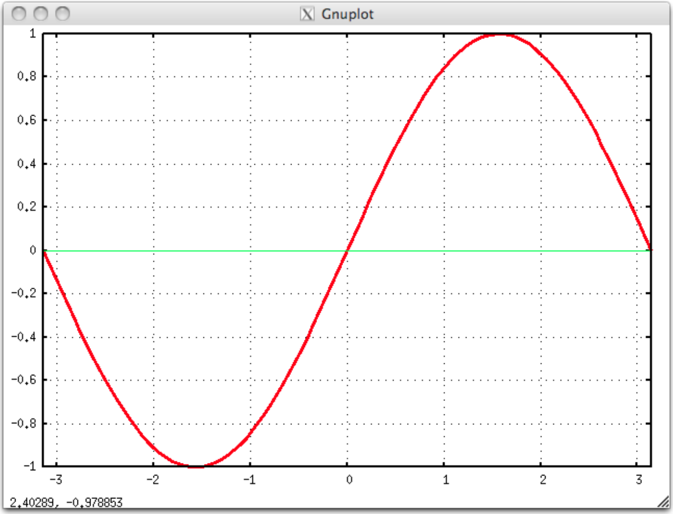
$f(x) = x^2$ is even.

Even and odd functions (4)



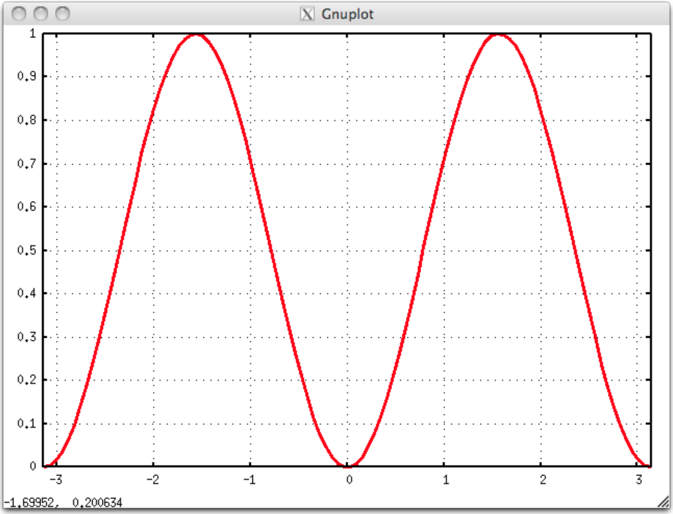
$f(x) = x^3$ is odd.

Even and odd functions (5)



$f(x) = \sin x$ is odd.

Even and odd functions (6)



$f(x) = (\sin x)^2$ is even.

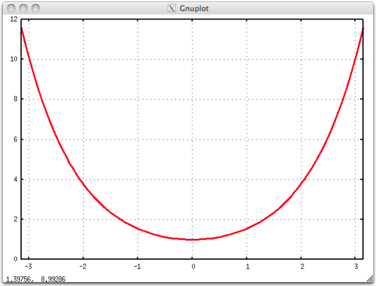
Even and odd functions (6)

Any function $f(x)$ can be decomposed as the sum of an even and an odd function :

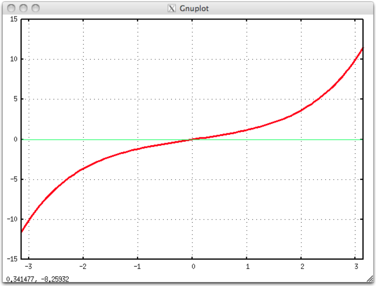
$$f(x) = \frac{1}{2}\left[f(x) + f(-x)\right] + \frac{1}{2}\left[f(x) - f(-x)\right]$$

Example :

$$e^x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = \cosh x + \sinh x$$



$f(x) = \cosh x$



$f(x) = \sinh x$

Even and odd functions

From the definition of even and odd functions, it naturally follows :

$$\int_{-l}^{l} f(x)dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd} \\ 2\int_0^l f(x)dx & \text{if } f(x) \text{ is even} \end{cases}$$

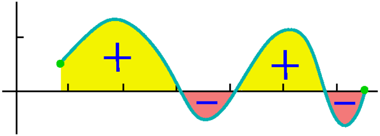
Even and odd functions

From the definition of even and odd functions, it naturally follows :

$$\int_{-l}^{l} f(x)dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd} \\ 2\int_0^l f(x)dx & \text{if } f(x) \text{ is even} \end{cases}$$

proof :

- think of an integral as a signed area calculation :



- split $\int_{-l}^{l}$ into $\int_{-l}^{0} + \int_0^l$ and use properties of even and odd functions

Even and odd functions (2)

For even or odd functions, the coefficients a_n and b_n simplify :

- If $f(x)$ is odd,

$$a_n = 0 \qquad\qquad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

- If $f(x)$ is even,

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx \qquad\qquad b_n = 0$$

An application to sound (1)

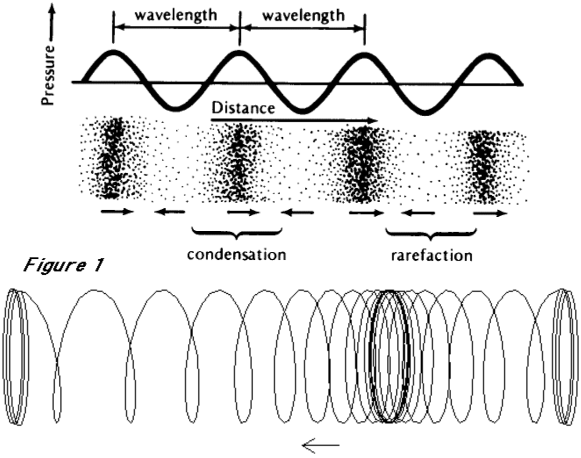


Figure 1

An application to sound (2)

The essential characteristics of a musical note

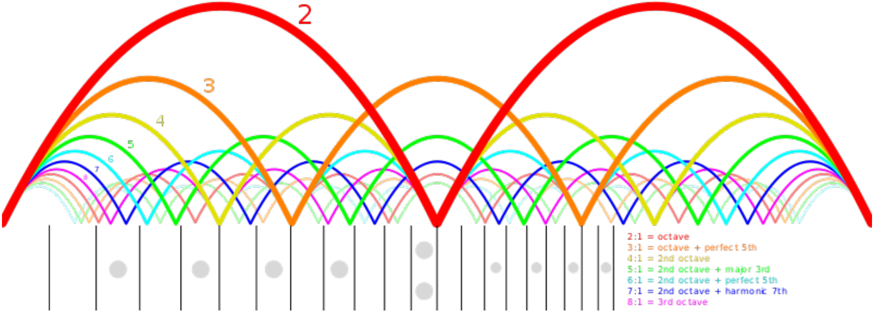
- The loudness of the note is measured by the magnitude of the changes in air pressure. This is controlled by how hard a piano key is pressed or how hard one blows on the mouthpiece of a saxophone.
- The pitch of the note is the frequency of repetition of the basic pressure pattern. More precisely, the frequency is the number of times the basic pattern is repeated per unit of time.
  - The frequencies of interest to us will be measured in cycles per second
  - One cycle per second is called a hertz in honor of Heinrich Hertz.
  - Human hearing is confined to frequencies that range roughly from 20 to 18,000 hertz.
- The timbre of the note includes those characteristics that enable us to tell a piano note from a violin note with the same loudness and pitch.

Maths, hairstyle and history (continued)

Heinrich Rudolf Hertz (1857-1894), German physicist

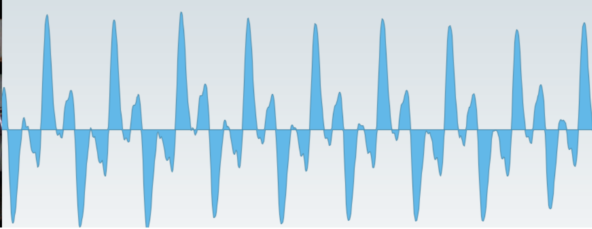
An application to sound (2) - guitar string spectrum

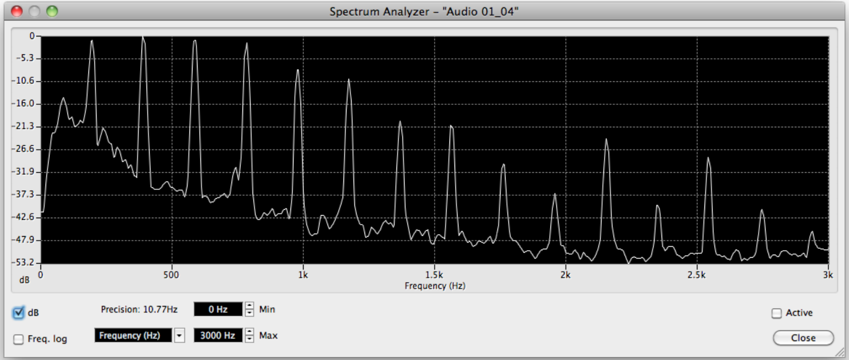
The nodes of natural harmonics are located at the following points along a guitar's neck.



An application to sound (3) - guitar string spectrum

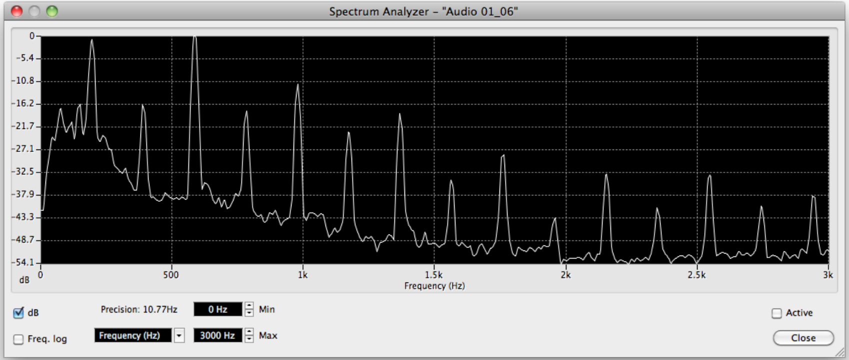
→ string played next to bridge





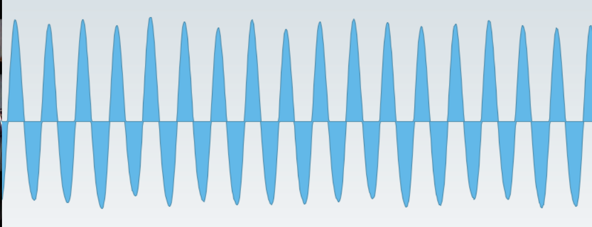
An application to sound (4) - guitar string spectrum

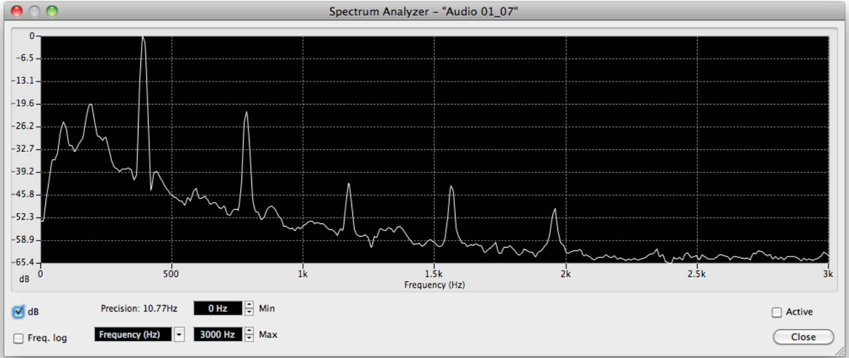
string played at 12th fret (half length)



An application to sound (5) - guitar string spectrum

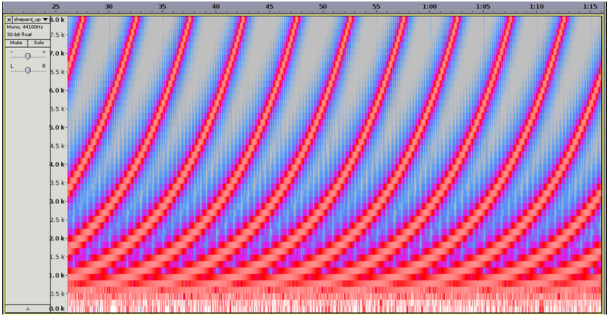
harmonic (placing a finger on the string when the string is driven)





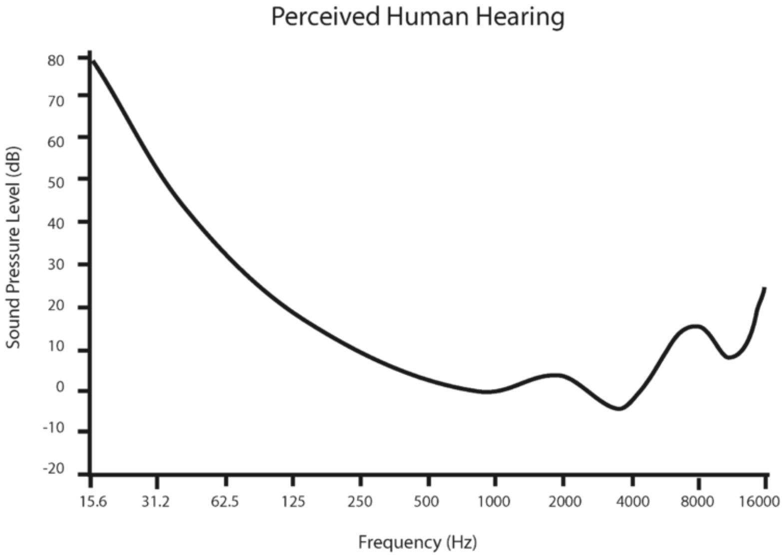
An application to sound (6)

- A Shepard tone, named after Roger Shepard, is a sound consisting of a superposition of sine waves separated by octaves.
- When played with the base pitch of the tone moving upward or downward, it is referred to as the Shepard scale.
- This creates the auditory illusion of a tone that continually ascends or descends in pitch, yet which ultimately seems to get no higher or lower.



http://en.wikipedia.org/wiki/Shepard_tone

An application to sound (7) - psychoacoustics



An application to sound (8) - mp3 encoding

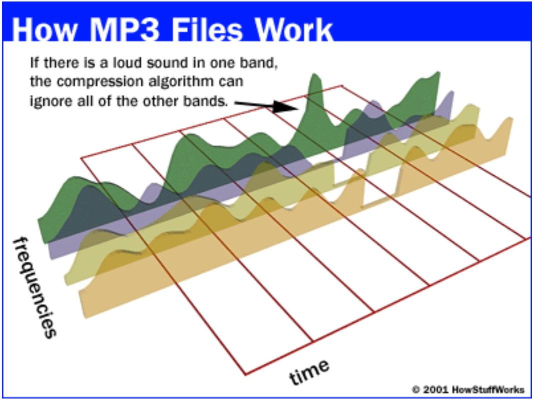
the mp3 format uses characteristics of the human ear to design the compression algorithm :

- There are certain sounds that the human ear cannot hear.
- There are certain sounds that the human ear hears much better than others.
- If there are two sounds playing simultaneously, we hear the louder one but cannot hear the softer one.

An application to sound (8) - mp3 encoding

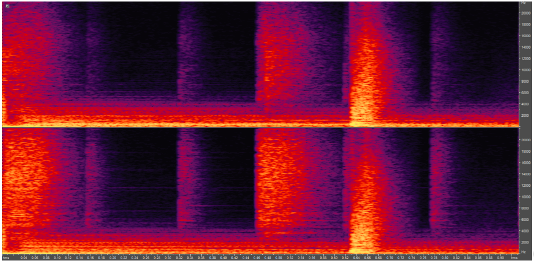
the mp3 format uses characteristics of the human ear to design the compression algorithm :

- There are certain sounds that the human ear cannot hear.
- There are certain sounds that the human ear hears much better than others.
- If there are two sounds playing simultaneously, we hear the louder one but cannot hear the softer one.

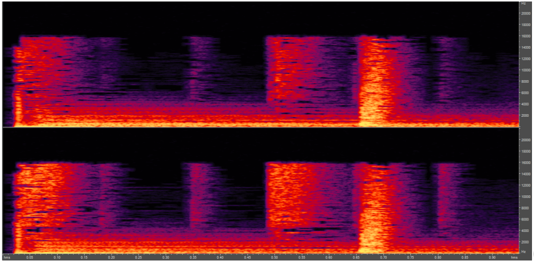


An application to sound (9) - mp3 encoding

Let us look at two spectrograms of a given piece of music

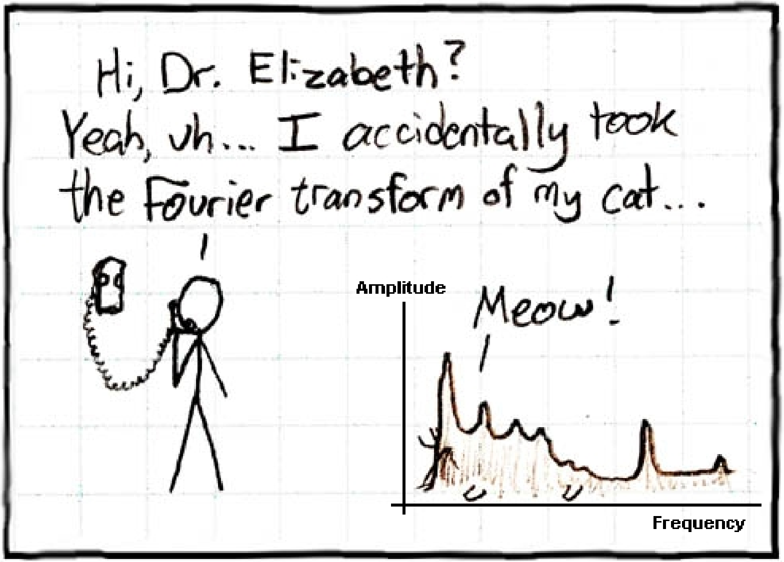


(wav file, original)



(mp3 file, 128kbps)

⇒ the mp3 encoding format is destructive.



http://xkcd.com/

Parseval's theorem

Let $f(x)$ be a function and its Fourier series write :

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Parseval's theorem

Let $f(x)$ be a function and its Fourier series write :

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

<u>Theorem</u> : The sum (or integral) of the square of a function is equal to the sum (or integral) of the square of its transform.

$$\text{The average of } [f(x)]^2 \text{ over a period } = (\frac{1}{2}a_0)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(b_n)^2$$

Parseval's theorem

Let $f(x)$ be a function and its Fourier series write :

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

<u>Theorem</u> : The sum (or integral) of the square of a function is equal to the sum (or integral) of the square of its transform.

$$\text{The average of } [f(x)]^2 \text{ over a period } = (\frac{1}{2}a_0)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(b_n)^2$$

If $f(x) \in \mathbb{C}$, the equality writes simply :

$$\text{The average of } |f(x)|^2 \text{ over a period } = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Parseval's theorem

Let $f(x)$ be a function and its Fourier series write :

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Theorem : The sum (or integral) of the square of a function is equal to the sum (or integral) of the square of its transform.

$$\text{The average of } [f(x)]^2 \text{ over a period } = (\frac{1}{2}a_0)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(b_n)^2$$

If $f(x) \in \mathbb{C}$, the equality writes simply :

$$\text{The average of } |f(x)|^2 \text{ over a period } = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Parseval's theorem is also called the completeness relation.

Maths, hairstyle and history (continued)

Marc-Antoine Parseval des Chênes (1755 - 1836), French mathematician

Parseval's theorem (3)

Example 1 : use Parseval's theorem with $f(x) = x^2$ to evaluate $\sum_{n=1}^{\infty} \frac{1}{n^4}$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2 dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{2\pi^4}{5}$$

The function $f(x)$ is even so $b_n = 0 \; \forall n$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(0x)\; dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^3}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\; dx = \frac{4(-1)^{n+1}}{n^2}$$

PT implies that

$$\frac{2\pi^4}{5} = \left(\frac{1}{2}\frac{2\pi^2}{3}\right)^2 + \sum_{n=1}^{\infty}\left(\frac{4(-1)^{n+1}}{n^2}\right)^2 = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\Rightarrow \qquad \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

Parseval's theorem (3)

Example 2 : We can use Parseval's theorem to find the sum of an infinite series.
Let us consider the function $f(x) = x$ of period 2 on $]-1, 1[$

$$f(x) = \frac{-i}{\pi}\left(e^{i\pi x} - e^{-i\pi x} - \frac{1}{2}e^{2i\pi x} + \frac{1}{2}e^{-2i\pi x} + \frac{1}{3}e^{3i\pi x} - \frac{1}{3}e^{-3i\pi x} + \cdots\right)$$

Let us find the average of $|f(x)|^2$ on $[-1 : 1]$:

$$\int_{-1}^{+1} |f(x)|^2 dx = \frac{1}{2}\int_{-1}^{+1} x^2 dx = \frac{1}{3}$$

By Parseval's theorem, this is equal to $\sum_{-\infty}^{\infty} |c_n|^2$:

$$\frac{1}{3} = \sum_{-\infty}^{\infty} |c_n|^2 = \frac{1}{\pi^2}(1 + 1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{9} + \frac{1}{9} + \cdots)$$

$$\Rightarrow 1 + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Fourier transforms (1)

- So far, we have been expanding *periodic* functions in series of sines, cosines, and complex exponentials.

Fourier transforms (1)

- So far, we have been expanding *periodic* functions in series of sines, cosines, and complex exponentials.
- **Question** : is it possible to represent a function which is not periodic by something analogous to Fourier series ?

Fourier transforms (1)

- So far, we have been expanding *periodic* functions in series of sines, cosines, and complex exponentials.
- **Question** : is it possible to represent a function which is not periodic by something analogous to Fourier series ?
- **Question** : Can we extend/modify the Fourier series to cover the case of a continuous spectrum of frequencies ?

Fourier transforms (2) - Definitions

Remember these ?

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/l} \qquad\qquad c_n = \frac{1}{2l}\int_{-l}^{+l} f(x) e^{-in\pi x/l} dx$$

The period of $f(x)$ is $2l$ and the frequencies are $n/(2l)$.

Fourier transforms (2) - Definitions

Remember these ?

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/l} \qquad\qquad c_n = \frac{1}{2l}\int_{-l}^{+l} f(x)e^{-in\pi x/l}dx$$

The period of $f(x)$ is $2l$ and the frequencies are $n/(2l)$.

Definition of Fourier transforms :

$$f(x) = \int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}d\alpha \qquad\qquad g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx$$

$g(\alpha)$ corresponds to c_n, α corresponds to n, and the integral corresponds to the discrete sum.
$g(\alpha)$ is called the Fourier transform of $f(x)$.
$f(x)$ is called the inverse Fourier transform of $g(\alpha)$.
The Fourier integral theorem states : if a function $f(x)$ satisfies the Dirichlet conditions on every finite interval, and if $\int_{\infty}^{\infty} |f(x)|dx$ is finite, then the definitions hereabove are correct.

Fourier transforms (2) - Fourier Sine/Cosine tansforms

We define $f_s(x)$ and $g_s(\alpha)$, a pair of fourier sine transforms representing *odd* functions by the equations

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g_s(\alpha) \sin(\alpha x) \, d\alpha$$

$$g_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_s(x) \sin(\alpha x) \, dx$$

We define $f_c(x)$ and $g_c(\alpha)$, a pair of fourier cosine transforms representing *even* functions by the equations

$$f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g_c(\alpha) \cos(\alpha x) \, d\alpha$$

$$g_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_c(x) \cos(\alpha x) \, dx$$

Fourier transforms (3)

Example 1 : Let us represent a nonperiodic functions as a Fourier integral
The function

$$f(x) = \begin{cases} 1, & -1 < x < 1 \\ 0, & |x| > 1 \end{cases}$$

Since the function is not periodic, it *cannot* be expanded in a Fourier series, since a Fourier series always represents a periodic function. Instead we compute $g(\alpha)$

$$g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx = \frac{1}{2\pi} \int_{-1}^{1} e^{-i\alpha x} dx = \frac{\sin \alpha}{\pi \alpha}$$

We have

$$f(x) = \int_{-\infty}^{\infty} g(\alpha) e^{i\alpha x} d\alpha = \int_{-\infty}^{\infty} \frac{\sin \alpha}{\pi \alpha} e^{i\alpha x} d\alpha = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin \alpha \cos \alpha x}{\alpha} d\alpha$$

since $(\sin \alpha)/\alpha$ is an even function.

Fourier transforms (3)

Example 2 :
We have established that

$$\frac{2}{\pi}\int_0^{\infty}\frac{\sin\alpha\cos\alpha x}{\alpha}d\alpha = \begin{cases} 1, & -1 < x < 1 \\ 0, & |x| > 1 \\ 1/2 & x = 1 \end{cases}$$

Where we have used the fact that the Fourier integral represents the midpoint of the jump in $f(x)$ at $|x| = 1$. If we let $x = 0$, we get

$$\int_0^{\infty}\frac{\sin\alpha}{\alpha}d\alpha = \frac{\pi}{2}$$

Parseval's theorem for Fourier integrals

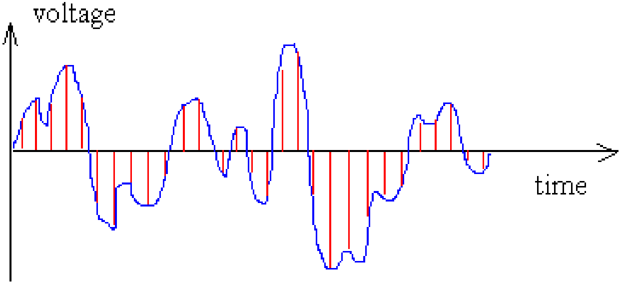
- Recall Parseval's theorem for fourier series :

$$\text{The average of } |f(x)|^2 \text{ over a period } = \sum_{n=-\infty}^{\infty} |c_n|^2$$

- this says that the total energy is equal to the sum of the energies associated with the various harmonics.
- Parseval's theorem for Fourier transforms writes as follows :

$$\int_{-\infty}^{\infty} |g(\alpha)|^2 d\alpha = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x)|^2 dx$$

Fun fact



CD-quality music recording is created by sampling the sound 44,100 times per second and storing each sample as a 16-bit binary number (twice as much for a stereo recording).
So an hour of stereo music is equivalent to $3,600 \times 44,100 \times 2 = 317,520,000$ samples or 635,040,000 bytes.