Differentiaal Vergelijkingen In de Aardwetenschappen

PDEs - chapt 13 - wave equation

C. Thieulot (c.thieulot@uu.nl)

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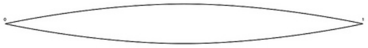
The wave equation (1)

- Consider a string attached at $x = 0$ and $x = L$



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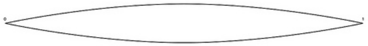

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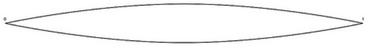

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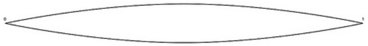

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The constant v depends on the tension and the linear density of the string. It is called the wave velocity.

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The solutions read

$$f(x) = \begin{cases} \sin kx \\ \cos kx \end{cases} \qquad g(t) = \begin{cases} \sin kvt \\ \cos kvt \end{cases}$$

The wave equation (3)

Recall that

- ν= frequency
- λ= wavelength
- $v = \lambda\nu$
- $\omega = 2\pi\nu$ =angular velocity
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so that the general solution looks like

$$y(x,t) = f(x)g(t) = \left\{ \begin{array}{c} \sin kx \\ \cos kx \end{array} \right\} \left\{ \begin{array}{c} \sin \omega t \\ \cos \omega t \end{array} \right\}$$

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then

$$y(x,t) = f(x)g(t) = \sin\frac{n\pi x}{L}\left\{\begin{array}{l}\sin\frac{n\pi vt}{L}\\ \cos\frac{n\pi vt}{L}\end{array}\right\}$$

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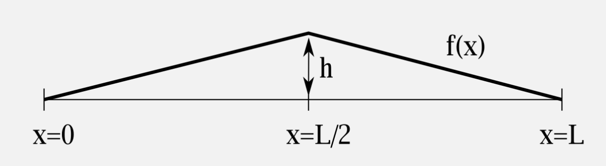
The solution will be a superposition of these basis functions :

$$y(x,t) = \sum_{n=1}^{\infty} b_n y_n(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{n\pi vt}{L}$$

The wave equation (6)

Initial condition : at $t = 0$, $y(x, 0) = f(x)$.

$$f(x) = \begin{cases} +\frac{2h}{L}x & 0 \leq x \leq L/2 \\ -\frac{2h}{L}x + 2h & L/2 \leq x \leq L \end{cases}$$



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$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

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$$\begin{aligned} b_n &= \frac{2}{L}\left\{\int_0^{L/2} \frac{2h}{L} x \sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} \frac{2h}{L}(L-x)\sin\frac{n\pi x}{L}\,dx\right\} \\ &= \ldots \end{aligned}$$

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Finally

$$y(x,t) = \frac{8h}{\pi^2}\left(\sin \frac{\pi x}{L} \cos \frac{\pi v t}{L} - \frac{1}{9} \sin \frac{3\pi x}{L} \cos \frac{3\pi v t}{L} + \cdots\right)$$