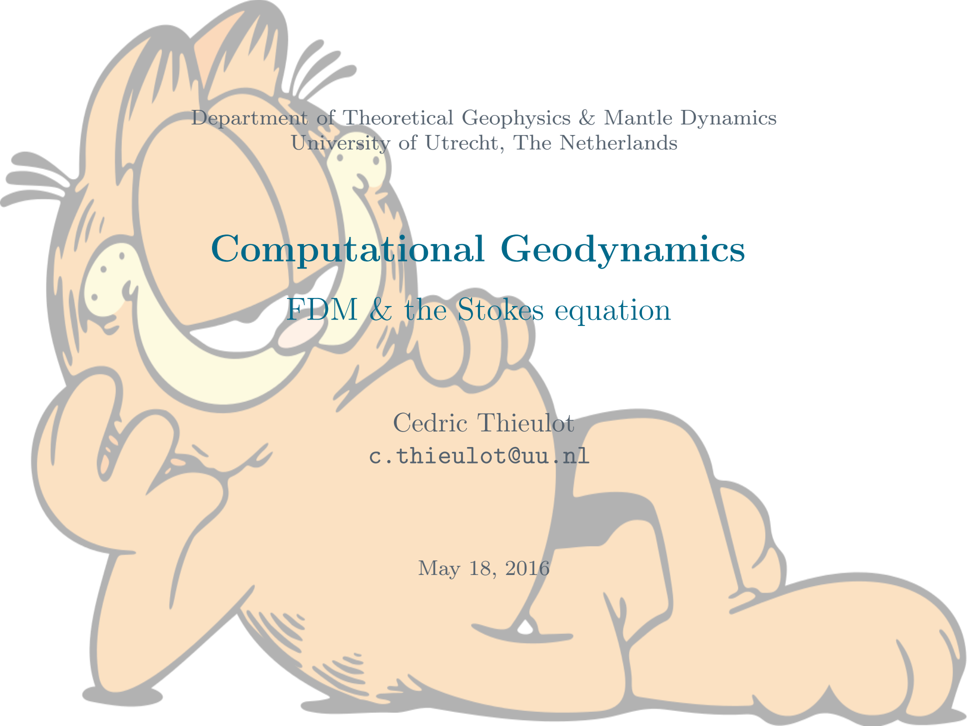

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Computational Geodynamics

FDM & the Stokes equation

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May 18, 2016

Content

What follows is valid for $\mu =$constant !

The Stokes eqs

Mass conservation equation for incompressible fluids:

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$$

Momentum conservation equation

$$-\boldsymbol{\nabla} p + \boldsymbol{\nabla} \cdot \boldsymbol{s} = \rho \boldsymbol{g} \qquad \text{with} \quad \boldsymbol{s} = 2\mu\dot{\boldsymbol{\epsilon}} = \mu(\boldsymbol{\nabla}\boldsymbol{v} + \nabla\boldsymbol{v}^T)$$

$$-\frac{\partial p}{\partial x} + \frac{\partial s_{xx}}{\partial x} + \frac{\partial s_{xy}}{\partial y} = \rho g_x$$

$$-\frac{\partial p}{\partial y} + \frac{\partial s_{xy}}{\partial x} + \frac{\partial s_{yy}}{\partial y} = \rho g_y$$

The Stokes eqs

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or,

$$-\frac{\partial p}{\partial x} + 2\mu\frac{\partial \dot{\epsilon}_{xx}}{\partial x} + 2\mu\frac{\partial \dot{\epsilon}_{xy}}{\partial y} = \rho g_x$$

$$-\frac{\partial p}{\partial y} + 2\mu\frac{\partial \dot{\epsilon}_{xy}}{\partial x} + 2\mu\frac{\partial \dot{\epsilon}_{yy}}{\partial y} = \rho g_y$$

The Stokes eqs
x-component

In 2D, we first explicitely write

$$\dot{\epsilon}_{xx} = \frac{\partial u}{\partial x} \qquad \dot{\epsilon}_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \dot{\epsilon}_{yy} = \frac{\partial v}{\partial y}$$

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Then

$$-\frac{\partial p}{\partial x} + 2\mu\frac{\partial \dot{\epsilon}_{xx}}{\partial x} + 2\mu\frac{\partial \dot{\epsilon}_{xy}}{\partial y} = \rho g_x$$

becomes

$$-\frac{\partial p}{\partial x} + 2\mu\frac{\partial^2 u}{\partial x^2} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) = \rho g_x$$

The Stokes eqs
x-component

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Using $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ to obtain $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$ we arrive at

$$-\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \rho g_x$$

The Stokes eqs

x, y-component in 2D

These are then the two coupled equations we need to solve

$$\boxed{-\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \rho g_x}$$

$$\boxed{-\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) = \rho g_y}$$

along with

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$$

The Stokes eqs

x, y-component in 2D

These are then the two coupled equations we need to solve

$$-\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \rho g_x$$

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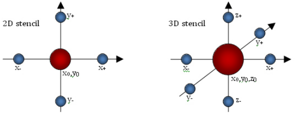
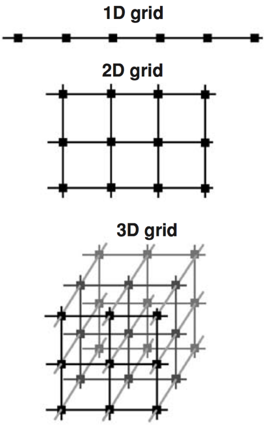
along with

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$



2D/3D stencils for Laplacian



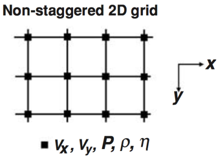


Fig. 7.5 Example of a non-staggered 2D numerical grid.

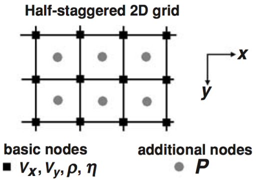


Fig. 7.6 Example of a half-staggered 2D numerical grid.

From Gerya, 2010

- Fully staggered grids are applied in 2D and consist of a combination of several types of nodal points having different geometrical positions.

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- Different variables are then defined at different nodal points.

Num. solutions of the Stokes eqs
Grids & stencils

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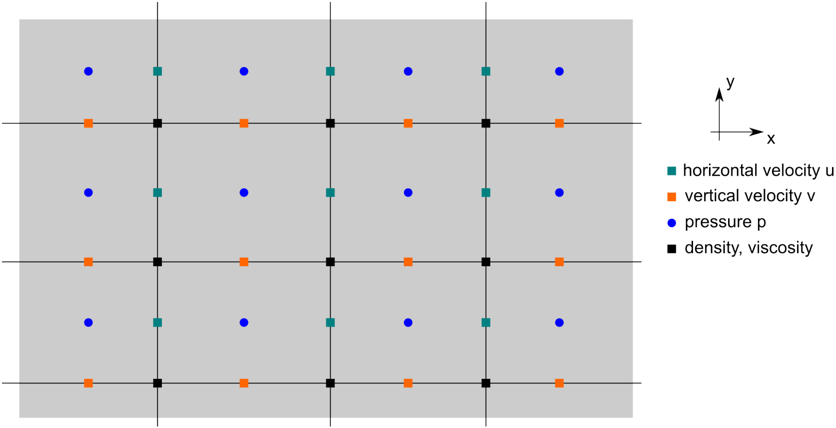
- Different variables are then defined at different nodal points.
- Different equations are also formulated at different nodal points.

Num. solutions of the Stokes eqs

Grids & stencils

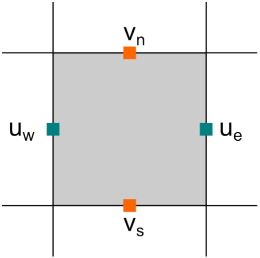
- Fully staggered grids are applied in 2D and consist of a combination of several types of nodal points having different geometrical positions.
- Different variables are then defined at different nodal points.
- Different equations are also formulated at different nodal points.
- Despite the apparent geometrical complexity, fully staggered grids are the most convenient choice for thermomechanical numerical problems with variable viscosity when finite differences are used for solving the Stokes and temperature equations.



Example of a fully staggered 2D numerical grid

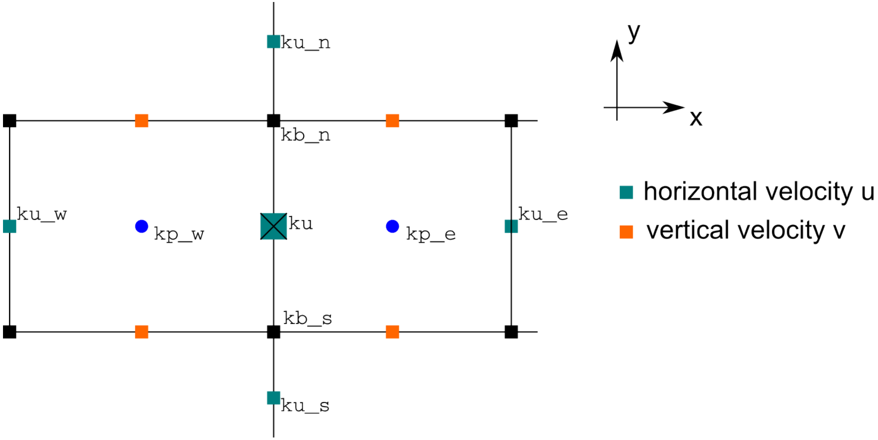
For incompressible flow:

$$\nabla \cdot \boldsymbol{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$



n~North
s~South
w~West
e~East

$$\frac{u_e - u_w}{h_x} + \frac{v_n - v_s}{h_y} = 0$$



ku_n
kb_n
ku_w
kp_w
ku
kp_e
ku_e
kb_s
ku_s
y
x
horizontal velocity u
vertical velocity v

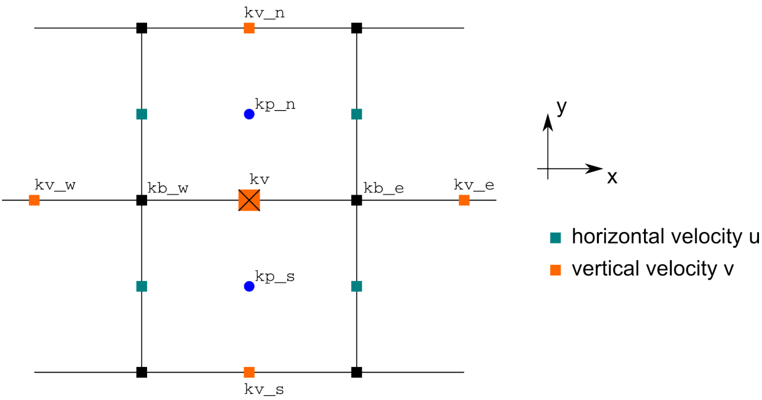
Num. solutions of the Stokes eqs

2D staggered grid, x-direction

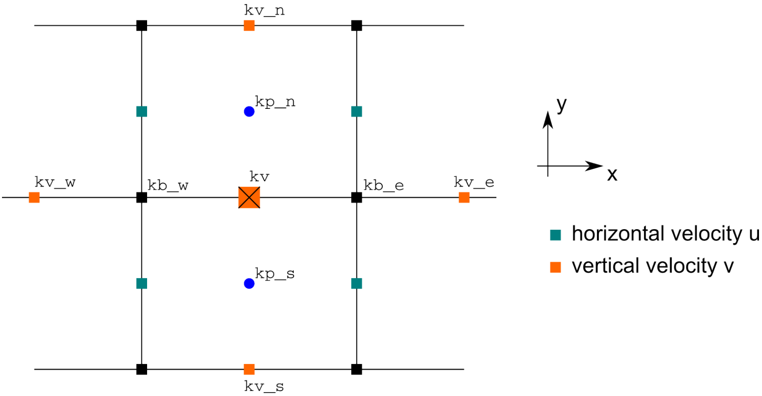
$$
\begin{aligned}
-\frac{\partial p}{\partial x}\bigg|_{\texttt{ku}} &= -\frac{p_{\texttt{kp\_e}} - p_{\texttt{kp\_w}}}{h_x} \\
\mu \frac{\partial^2 u}{\partial x^2}\bigg|_{\texttt{ku}} &= \mu \frac{u_{\texttt{ku\_e}} - 2u_{\texttt{ku}} + u_{\texttt{ku\_w}}}{h_x^2} \\
\mu \frac{\partial^2 u}{\partial y^2}\bigg|_{\texttt{ku}} &= \mu \frac{u_{\texttt{ku\_n}} - 2u_{\texttt{ku}} + u_{\texttt{ku\_s}}}{h_y^2} \\
\rho g_x\big|_{\texttt{ku}} &= \frac{\rho_{\texttt{kb\_n}} + \rho_{\texttt{kb\_s}}}{2} g_x
\end{aligned}
$$

Num. solutions of the Stokes eqs

2D staggered grid, y-direction



Num. solutions of the Stokes eqs
2D staggered grid, y-direction



- Similar expressions can be written for the $z-$component of the momentum equation.
- Not difficult, only careful bookkeeping

Num. solutions of the Stokes eqs

2D staggered grid, y-direction

$$
\begin{aligned}
-\left.\frac{\partial p}{\partial y}\right|_{\texttt{kv}} &= -\frac{p_{\texttt{kp\_n}} - p_{\texttt{kp\_s}}}{h_y} \\
\mu \left.\frac{\partial^2 v}{\partial x^2}\right|_{\texttt{kv}} &= \mu \frac{v_{\texttt{kv\_e}} - 2 v_{\texttt{kv}} + v_{\texttt{kv\_w}}}{h_x^2} \\
\mu \left.\frac{\partial^2 v}{\partial y^2}\right|_{\texttt{kv}} &= \mu \frac{v_{\texttt{kv\_n}} - 2 v_{\texttt{kv}} + v_{\texttt{kv\_s}}}{h_y^2} \\
\rho g_y \big|_{\texttt{kv}} &= \frac{\rho_{\texttt{kb\_s}} + \rho_{\texttt{kb\_n}}}{2} g_y
\end{aligned}
$$

Mechanical boundary conditions depend on the type of numerical problem which is studied.
The following boundary conditions are often used in geomodelling:

- free slip
- no slip
- free surface
- fast erosion
- infinity-like (external free slip, external no slip, Winkler basement)
- prescribed velocity (moving boundary)
- periodic
- combined conditions

- free slip b.c.: the normal velocity component on the boundary is zero and the two other components do not change across the boundary (this condition also implies zero shear strain rates and stresses along the boundary). For example, for the boundary orthogonal to the x axis, the free slip condition is formulated as follows

$$u = 0 \qquad\qquad \frac{\partial v}{\partial x} = \frac{\partial w}{\partial x} = 0$$

- A no slip condition requires all velocity components on the boundary to be zero, i.e.

$$u = v = w = 0$$

- A free surface condition requires both shear and normal stresses at the boundary to be zero

$$s_{ij} = 0$$

Num. solutions of the Stokes eqs

Boundary conditions

- The prescribed velocity condition implies non-zero velocity at a model boundary. When velocity is prescribed orthogonal to the boundary (inward/outward flow), then a compensating outward/inward velocity should be prescribed on the other model boundary(ies) in order to insure mass conservation in the model.

Geochemistry Geophysics Geosystems G3 GURNIS ET AL.: EVOLVING FORCE BALANCE 10.1029/2003GC000681

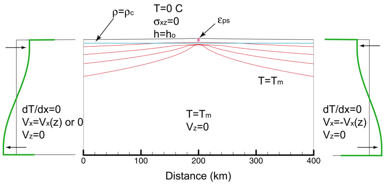


Figure 5. Model setup showing boundary and initial conditions. The initial temperature field is shown with 300°C isotherms in red for Case 15 (as an example). The velocity boundary conditions are shown schematically off to the left and the right of the domain. The depth scale is the same as the horizontal.

- Periodic boundary conditions are typically established for paired parallel lateral boundaries of a model and prescribe that all material properties as well as pressure and velocity fields at both sides of each boundary are identical. From a physical point of view, this implies that these two boundaries are open and that flow leaving the model through one boundary immediately re-enters through the opposite side. This condition is often used in mantle convection modelling to simulate part of a spherical/cylindrical shell with a convecting mantle (or mimic it, in Cartesian coordinates).
- Combined conditions represent a mixture between several types of boundary conditions.

- All of the described boundary conditions can be time dependent.

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- This could particularly imply that the physical location of the boundary condition may be a function of time.

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- Boundary conditions can also be applied inside the model.

Num. solutions of the Stokes eqs

Boundary conditions

implementation free and no slip for staggered grid p94

Num. solutions of the Stokes eqs

Indexing of unknowns

- Another very important issue, in relation to solving the Stokes and continuity equations on a fully staggered grid, is the indexing (numbering) of the unknowns.

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- This is particularly relevant when the system of linear equations (global matrix) formulated with finite differences is solved with Gaussian elimination.

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- This is a somewhat boring subject but it is extremely important to understand it properly. (Remember, 90% of the bugs in your code are made with the indexing).

Num. solutions of the Stokes eqs

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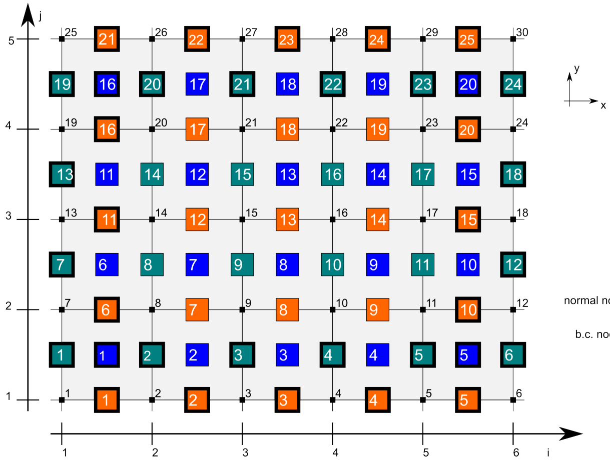
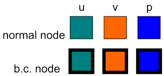
- Both the possibility of obtaining the solution and the amount of computational work will strongly depend on the method used to index the unknowns (p, u and v) on the staggered grid.
- One of the (optimal?) ways of numbering is illustrated hereafter:

Num. solutions of the Stokes eqs

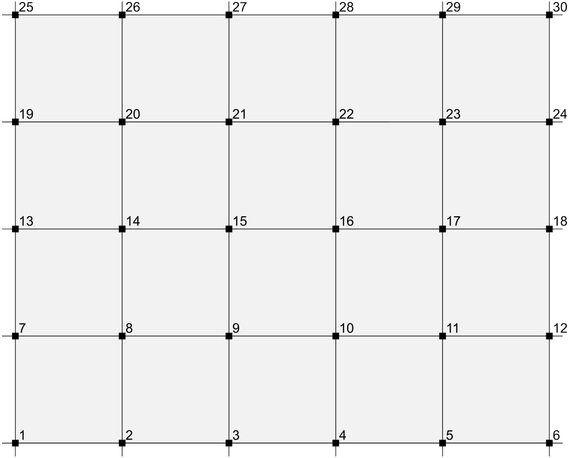
A simple 6x5 grid



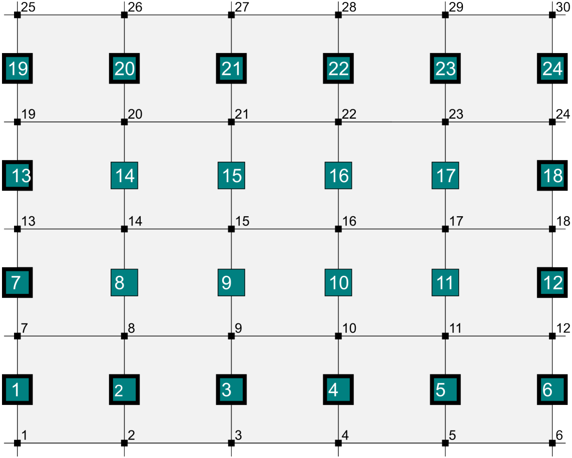


nnx=6
nny=5
30 nodes
90 unknowns
68 real unknowns

Background grid

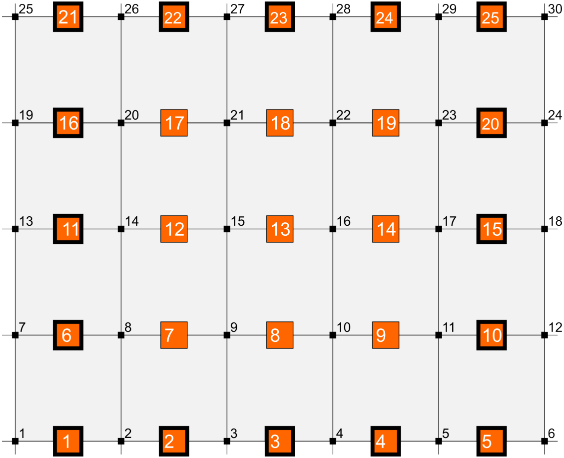


Background grid + u grid

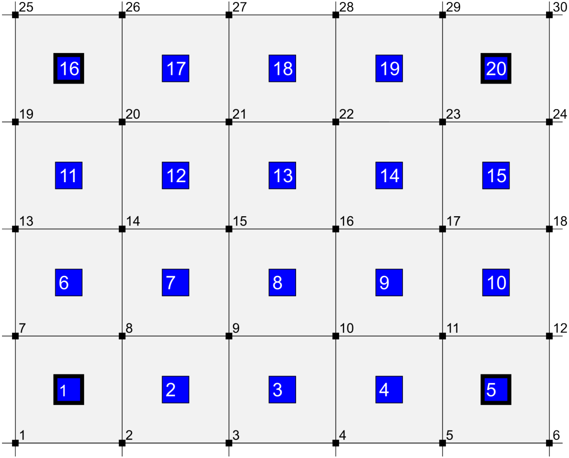


Num. solutions of the Stokes eqs
A simple 6x5 grid

Background grid + v grid



Background grid + p grid



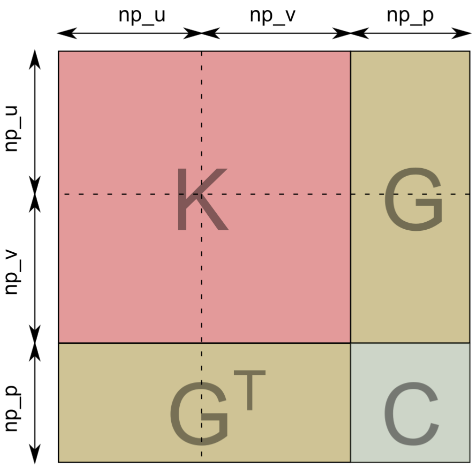
Num. solutions of the Stokes eqs
Matrix structure

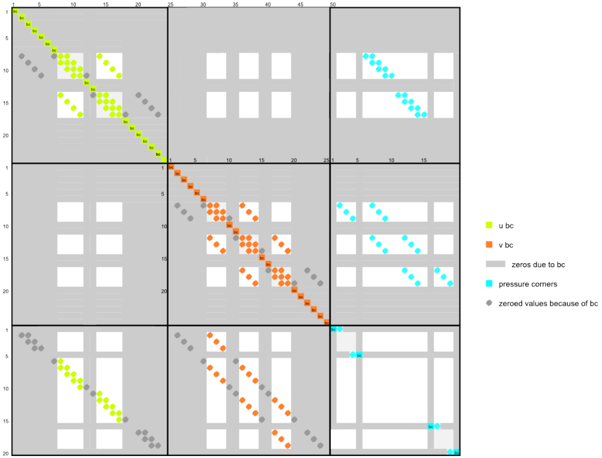
The discretisation of the Stokes equations by means of the FDM or FEM yield a discrete system of equations which takes the form:

$$\begin{pmatrix} K & G \\ G^T & 0 \end{pmatrix}$$

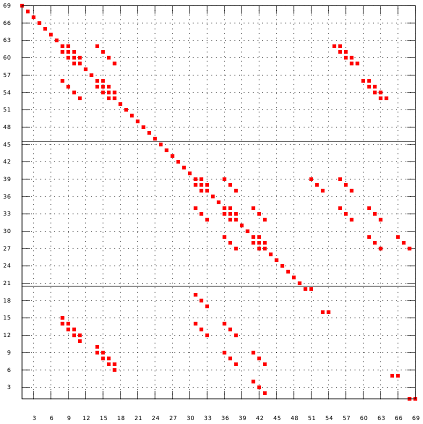
Num. solutions of the Stokes eqs

Matrix structure



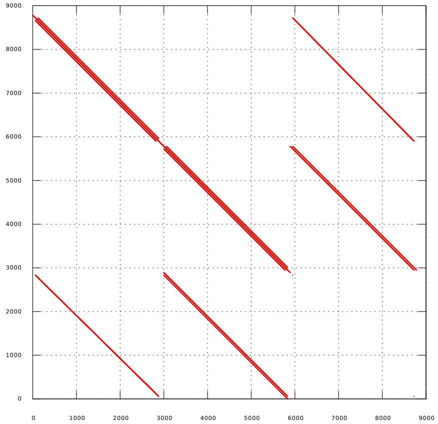

u bc
v bc
zeros due to bc
pressure corners
zeroed values because of bc

Num. solutions of the Stokes eqs
Recovered matrix structure



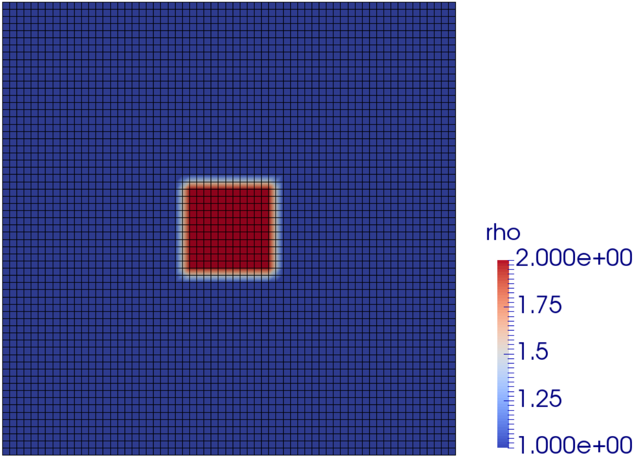
Num. solutions of the Stokes eqs

Matrix structure for 60x50 grid

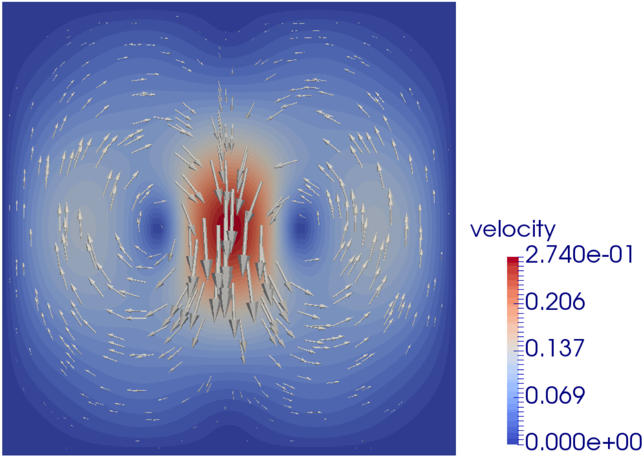


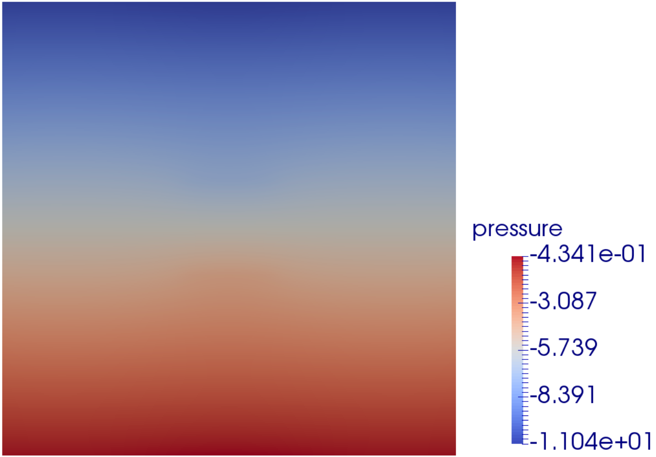
Results

The sinking cube



Results
The sinking cube




pressure
-4.341e-01
-3.087
-5.739
-8.391
-1.104e+01

Results

The sinking cube