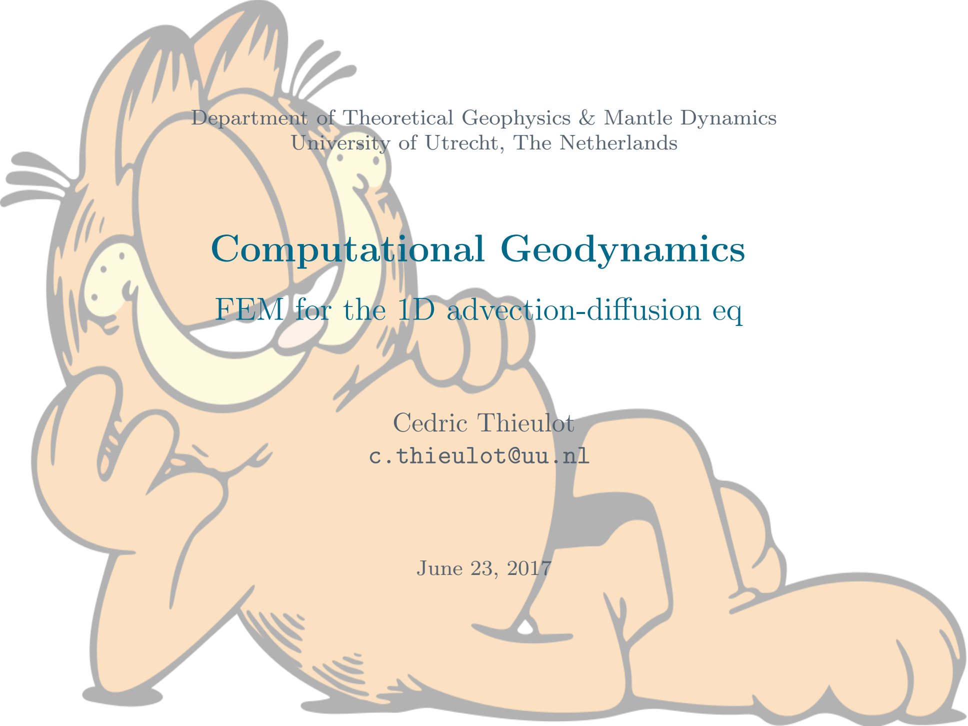

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Computational Geodynamics

FEM for the 1D advection-diffusion eq

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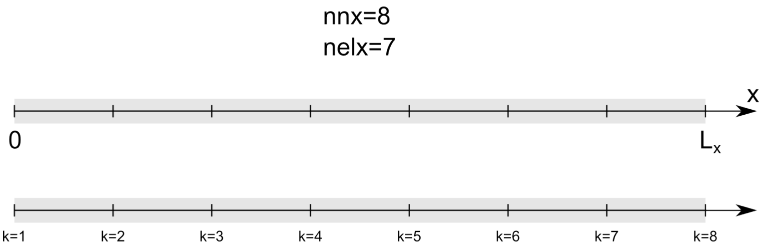
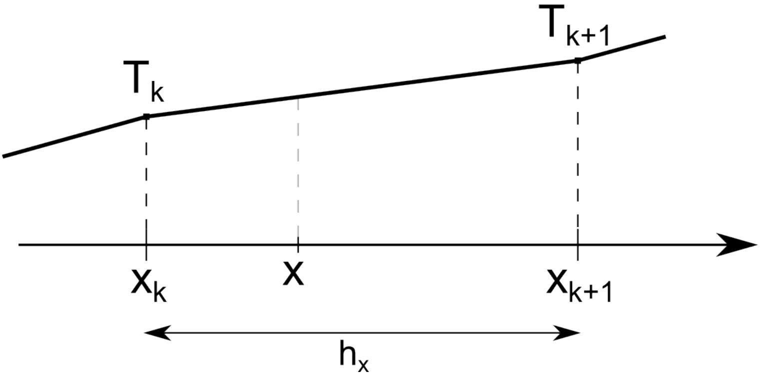
Content



- domain Ω of length L_x
- 1D grid, *nnx* nodes, *nelx* elements

FEM in 1D

Zoom on one element



- We start with the 1D advection-diffusion equation

$$\rho c_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} \right) = \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + H$$

- This is the strong form of the ODE to solve.
- I multiply this equation by a function $f(x)$ and integrate it over Ω:

$$\int_\Omega f(x) \rho c_p \frac{\partial T}{\partial t} dx + \int_\Omega f(x) \rho c_p u \frac{\partial T}{\partial x} dx = \int_\Omega f(x) \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) dx + \int_\Omega f(x) H dx$$

- I integrate the r.h.s. by parts:

$$\int_\Omega f(x)\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx = \left[f(x)k\frac{\partial T}{\partial x}\right]_{\partial\Omega} - \int_\Omega \frac{\partial f}{\partial x}k\frac{\partial T}{\partial x}dx$$

- Assuming there is no heat flux prescribed on the boundary (i.e. $q_x = -k\partial T/\partial x = 0$), then:

$$\int_\Omega f(x)\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx = -\int_\Omega \frac{\partial f}{\partial x}k\frac{\partial T}{\partial x}dx$$

FEM in 1D
From the strong form to the weak form

We then obtain the weak form of the diffusion equation in 1D:

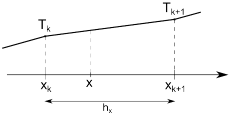
$$\int_\Omega f(x)\rho c_p \frac{\partial T}{\partial t} dx + \int_\Omega f(x)\rho c_p u \frac{\partial T}{\partial x} dx + \int_\Omega \frac{\partial f}{\partial x} k \frac{\partial T}{\partial x} dx = \int_\Omega f(x) H dx$$

We then use the additive property of the integral:

$$\int_\Omega \cdots = \sum_{elts} \int_{\Omega_e} \cdots$$

so that

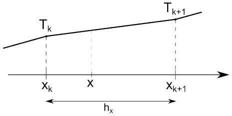
$$\sum_{elts} \left(\underbrace{\int_{\Omega_e} f \rho c_p \frac{\partial T}{\partial t} dx}_{\Lambda_f^e} + \underbrace{\int_{\Omega_e} f \rho c_p u \frac{\partial T}{\partial x} dx}_{\Sigma_f^e} + \underbrace{\int_{\Omega_e} \frac{\partial f}{\partial x} k \frac{\partial T}{\partial x} dx}_{\Upsilon_f^e} - \underbrace{\int_{\Omega_e} f H dx}_{\Omega_f^e} \right) = 0$$



In the element, we have seen that the temperature can be written:

$$T(x) = N_k(x)T_k + N_{k+1}(x)T_{k+1}$$

FEM in 1D
Shape functions



In the element, we have seen that the temperature can be written:

$$T(x) = N_k(x)T_k + N_{k+1}(x)T_{k+1}$$

In the previous presentation we have computed Λ_f^e and Υ_f^e. Let us now turn to Σ_f^e and Ω_f^e.

FEM in 1D

.

$$\Sigma_f^e = \int_{x_k}^{x_{k+1}} f(x) \rho c_p u \frac{\partial T}{\partial x} dx$$

FEM in 1D

.

$$
\begin{aligned}
\Sigma_f^e &= \int_{x_k}^{x_{k+1}} f(x)\rho c_p u \frac{\partial T}{\partial x} dx \\
&= \int_{x_k}^{x_{k+1}} f(x)\rho c_p u \frac{\partial [N_k(x) T_k + N_{k+1}(x) T_{k+1}]}{\partial x} dx
\end{aligned}
$$

FEM in 1D

.

$$
\begin{aligned}
\Sigma_f^e &= \int_{x_k}^{x_{k+1}} f(x)\rho c_p u \frac{\partial T}{\partial x} dx \\
&= \int_{x_k}^{x_{k+1}} f(x)\rho c_p u \frac{\partial [N_k(x) T_k + N_{k+1}(x) T_{k+1}]}{\partial x} dx \\
&= \int_{x_k}^{x_{k+1}} f(x)\rho c_p u \frac{\partial N_k}{\partial x} T_k dx + \int_{x_k}^{x_{k+1}} f(x)\rho c_p u \frac{\partial N_{k+1}}{\partial x} T_{k+1} dx
\end{aligned}
$$

FEM in 1D

.

$$
\begin{aligned}
\Sigma_f^e &= \int_{x_k}^{x_{k+1}} f(x)\rho c_p u \frac{\partial T}{\partial x} dx \\
&= \int_{x_k}^{x_{k+1}} f(x)\rho c_p u \frac{\partial [N_k(x) T_k + N_{k+1}(x) T_{k+1}]}{\partial x} dx \\
&= \int_{x_k}^{x_{k+1}} f(x)\rho c_p u \frac{\partial N_k}{\partial x} T_k dx + \int_{x_k}^{x_{k+1}} f(x)\rho c_p u \frac{\partial N_{k+1}}{\partial x} T_{k+1} dx \\
&= \left(\int_{x_k}^{x_{k+1}} f(x)\rho c_p u \frac{\partial N_k}{\partial x} dx \right) T_k + \left(\int_{x_k}^{x_{k+1}} f(x)\rho c_p u \frac{\partial N_{k+1}}{\partial x} dx \right) T_{k+1}
\end{aligned}
$$

FEM in 1D

.

- Taking $f(x) = N_k(x)$ and omitting '(x)' in the rhs:

$$\Sigma_{N_k} = \left(\int_{x_k}^{x_{k+1}} \rho c_p u N_k \frac{\partial N_k}{\partial x} dx \right) T_k + \left(\int_{x_k}^{x_{k+1}} \rho c_p u N_k \frac{\partial N_{k+1}}{\partial x} dx \right) T_{k+1}$$

FEM in 1D

- Taking $f(x) = N_k(x)$ and omitting '(x)' in the rhs:

$$\Sigma_{N_k} = \left(\int_{x_k}^{x_{k+1}} \rho c_p u N_k \frac{\partial N_k}{\partial x} dx \right) T_k + \left(\int_{x_k}^{x_{k+1}} \rho c_p u N_k \frac{\partial N_{k+1}}{\partial x} dx \right) T_{k+1}$$

- Taking $f(x) = N_{k+1}(x)$ and omitting '(x)' in the rhs:

$$\Sigma_{N_{k+1}} = \left(\int_{x_k}^{x_{k+1}} \rho c_p u N_{k+1} \frac{\partial N_k}{\partial x} dx \right) T_k + \left(\int_{x_k}^{x_{k+1}} \rho c_p u N_{k+1} \frac{\partial N_{k+1}}{\partial x} dx \right) T_{k+}$$

FEM in 1D

.

$$\left(\begin{array}{c}\Sigma_{N_k}\\ \\ \Sigma_{N_{k+1}}\end{array}\right)=\left(\begin{array}{cc}\int_{x_k}^{x_{k+1}}\rho c_p u \mathbf{N}_k \frac{\partial \mathbf{N}_k}{\partial x}dx & \int_{x_k}^{x_{k+1}}\rho c_p u \mathbf{N}_k \frac{\partial \mathbf{N}_{k+1}}{\partial x}dx \\ \\ \int_{x_k}^{x_{k+1}}\rho c_p u \mathbf{N}_{k+1} \frac{\partial \mathbf{N}_k}{\partial x}dx & \int_{x_k}^{x_{k+1}}\rho c_p u \mathbf{N}_{k+1} \frac{\partial \mathbf{N}_{k+1}}{\partial x}dx\end{array}\right)\cdot\left(\begin{array}{c}T_k\\ \\ T_{k+1}\end{array}\right)$$

or,

FEM in 1D

.

$$\left(\begin{array}{c} \Sigma_{N_k} \\ \\ \Sigma_{N_{k+1}} \end{array}\right) = \left(\begin{array}{cc} \int_{x_k}^{x_{k+1}} \rho c_p u N_k \frac{\partial N_k}{\partial x} dx & \int_{x_k}^{x_{k+1}} \rho c_p u N_k \frac{\partial N_{k+1}}{\partial x} dx \\ \\ \int_{x_k}^{x_{k+1}} \rho c_p u N_{k+1} \frac{\partial N_k}{\partial x} dx & \int_{x_k}^{x_{k+1}} \rho c_p u N_{k+1} \frac{\partial N_{k+1}}{\partial x} dx \end{array}\right) \cdot \left(\begin{array}{c} T_k \\ \\ T_{k+1} \end{array}\right)$$

or,

$$\left(\begin{array}{c} \Sigma_{N_k} \\ \\ \Sigma_{N_{k+1}} \end{array}\right) = \left[\int_{x_k}^{x_{k+1}} \rho c_p u \left(\begin{array}{cc} N_k \frac{\partial N_k}{\partial x} & N_k \frac{\partial N_{k+1}}{\partial x} \\ \\ N_{k+1} \frac{\partial N_k}{\partial x} & N_{k+1} \frac{\partial N_{k+1}}{\partial x} \end{array}\right) dx \right] \cdot \left(\begin{array}{c} T_k \\ \\ T_{k+1} \end{array}\right)$$

Finally, we have already defined the vectors

$$\boldsymbol{N}^T = \begin{pmatrix} N_k(x) \\ \\ N_{k+1}(x) \end{pmatrix} \qquad \boldsymbol{B}^T = \begin{pmatrix} \frac{\partial N_k}{\partial x} \\ \\ \frac{\partial N_{k+1}}{\partial x} \end{pmatrix} \qquad \boldsymbol{T}^e = \begin{pmatrix} T_k \\ \\ T_{k+1} \end{pmatrix}$$

so that

$$\begin{pmatrix} \Sigma_{N_k} \\ \\ \Sigma_{N_{k+1}} \end{pmatrix} = \left(\int_{x_k}^{x_{k+1}} \boldsymbol{N}^T \rho c_p u \boldsymbol{B} dx \right) \cdot \boldsymbol{T}^e$$

Prove that

$$\boldsymbol{K}_a^e = \rho c_p u \left(\begin{array}{cc} -1/2 & 1/2 \\ \\ -1/2 & 1/2 \end{array} \right)$$

FEM in 1D

Let us now look at the source term:

$$\Omega_f^e = \int_{x_k}^{x^{k+1}} f(x)H(x)dx$$

- Taking $f(x) = N_k(x)$

$$\Omega_{N_k} = \int_{x_k}^{x^{k+1}} N_k(x)H(x)dx$$

FEM in 1D

Let us now look at the source term:

$$\Omega_f^e = \int_{x_k}^{x^{k+1}} f(x) H(x) dx$$

- Taking $f(x) = N_k(x)$

$$\Omega_{N_k} = \int_{x_k}^{x^{k+1}} N_k(x) H(x) dx$$

- Taking $f(x) = N_{k+1}(x)$

$$\Omega_{N_{k+1}} = \int_{x_k}^{x^{k+1}} N_{k+1}(x) H(x) dx$$

FEM in 1D

$$\left(\begin{array}{c} \Omega_{N_k} \\ \\ \Omega_{N_{k+1}} \end{array}\right) = \left(\begin{array}{c} \int_{x_k}^{x^{k+1}} N_k(x) H(x) dx \\ \\ \int_{x_k}^{x^{k+1}} N_{k+1}(x) H(x) dx \end{array}\right)$$

$$\left(\begin{array}{c}\Omega_{N_k}\\ \\ \Omega_{N_{k+1}}\end{array}\right)=\left(\begin{array}{c}\int_{x_k}^{x^{k+1}} N_k(x)H(x)dx\\ \\ \int_{x_k}^{x^{k+1}} N_{k+1}(x)H(x)dx\end{array}\right)$$

or,

$$\left(\begin{array}{c}\Omega_{N_k}\\ \\ \Omega_{N_{k+1}}\end{array}\right)=\left[\int_{x_k}^{x^{k+1}}\left(\begin{array}{c}N_k(x)H(x)\\ \\ N_{k+1}(x)H(x)\end{array}\right)dx\right]$$

so that

$$\left(\begin{array}{c}\Omega_{N_k}\\ \\ \Omega_{N_{k+1}}\end{array}\right)=\left(\int_{x_k}^{x^{k+1}}\boldsymbol{N}^T H(x)dx\right)$$

FEM in 1D

The weak form discretised over 1 element becomes

$$\underbrace{\left(\int_{x_k}^{x_{k+1}} \boldsymbol{N}^T \rho c_p \boldsymbol{N} dx \right)}_{\boldsymbol{M}^e} \cdot \dot{\boldsymbol{T}}^e + \underbrace{\left(\int_{x_k}^{x_{k+1}} \boldsymbol{N}^T \rho c_p u \boldsymbol{B} dx \right)}_{\boldsymbol{K}_a^e} \cdot \boldsymbol{T}^e$$

$$+ \underbrace{\left(\int_{x_k}^{x_{k+1}} \boldsymbol{B}^T k \boldsymbol{B} dx \right)}_{\boldsymbol{K}_d^e} \cdot \boldsymbol{T}^e = \underbrace{\left(\int_{x_k}^{x_{k+1}} \boldsymbol{N}^T H(x) dx \right)}_{\boldsymbol{F}^e}$$

or,

$$\boxed{\boldsymbol{M}^e \cdot \dot{\boldsymbol{T}}^e + (\boldsymbol{K}_d^e + \boldsymbol{K}_a^e) \cdot \boldsymbol{T}^e = \boldsymbol{F}^e}$$

or,

$$\boxed{\boldsymbol{M}^e \cdot \frac{\partial \boldsymbol{T}^e}{\partial t} + (\boldsymbol{K}_a^e + \boldsymbol{K}_d^e) \cdot \boldsymbol{T}^e = \boldsymbol{F}^e}$$

FEM in 1D

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Appendix A. FEM formulation of equations

The Galerkin finite element equation corresponding to Eq. (12) is

$$(\boldsymbol{K}_\mu + \boldsymbol{K}_\lambda) \cdot \boldsymbol{v} = \boldsymbol{B}$$

with

$$\boldsymbol{K}_\mu = \int_\Omega \boldsymbol{B}^T \cdot \boldsymbol{D}_\mu \cdot \boldsymbol{B} d\Omega$$

$$\boldsymbol{K}_\lambda = \int_\Omega \boldsymbol{B}^T \cdot \boldsymbol{D}_\lambda \cdot \boldsymbol{B} d\Omega$$

$$\boldsymbol{B} = \int_\Omega \boldsymbol{N}^T \rho \boldsymbol{g} d\Omega$$

$$\boldsymbol{D}_\mu^{2D} = \mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \boldsymbol{D}_\lambda^{2D} = \lambda \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and where $\boldsymbol{N}$ is the vector of shape functions, and $\boldsymbol{B}$ is the matrix of spatial derivatives of the shape functions.

The finite element equation corresponding to the heat transfer equation is

$$\boldsymbol{M}_c \cdot \frac{\partial \boldsymbol{T}}{\partial t} + (\boldsymbol{K}_a + \boldsymbol{K}_d) \cdot \boldsymbol{T} = \boldsymbol{F}$$

where

$$\boldsymbol{M}_c = \int_\Omega \boldsymbol{N}^T \rho c_P \boldsymbol{N} d\Omega$$

$$\boldsymbol{K}_a = \int_\Omega (\boldsymbol{N}^*)^T \rho c_P \boldsymbol{v} \cdot \boldsymbol{B} d\Omega \qquad \text{(A.1)}$$

$$\boldsymbol{K}_d = \int_\Omega \boldsymbol{B}^T k \boldsymbol{B} d\Omega$$

$$\boldsymbol{F} = \int_\Omega \boldsymbol{N}^T H d\Omega$$

where $\boldsymbol{T}$ is the vector of the nodal temperatures.

In the case where advection dominates over diffusion, the standard Galerkin approach of the advection term leads to problematic oscillations, and a stabilisation scheme is needed. A streamline-upwind Petrov–Galerkin (SUPG) method is therefore implemented, which translates in the modified $\boldsymbol{N}^*$ term in Eq. (A.1):

$$(\boldsymbol{N}^*)^T = \boldsymbol{N}^T + \tau \boldsymbol{v} \cdot \boldsymbol{B}$$

where τ a dimensionless parameter. The case $\tau = 0$ is equivalent to the Bubnov–Galerkin method. The choice of the parameter τ in the context of FEM stabilisation schemes is discussed in Tezduyar and Osawa (2000) and is calculated as follows:

$$\tau = \left(\frac{1}{(\tau_1)^r} + \frac{1}{(\tau_2)^r} + \frac{1}{(\tau_3)^r} \right)^{-1/r}$$

with often $r = 1$ or $r = 1/2$ and

$$\tau_1 = \frac{h}{2|\boldsymbol{v}|}, \quad \tau_2 = \theta dt, \quad \tau_3 = \frac{h^2 \rho c_P}{k} \qquad \text{(A.2)}$$

where h is a measure of the element size and θ is related to the time discretisation scheme ($\theta = 1/2$ in this case as it corresponds to the implemented mid-point implicit scheme, see for instance Braun (2003)). In order to illustrate the beneficial aspect of the SUPG scheme, let us look at the standard problem of the one-dimensional advection of a scalar field containing a steep front (diffusion and source terms are null). In Fig. 19 is shown the analytical initial scalar field. It is a challenging benchmark as the numerical treatment of the advection of such a discontinuity often leads to non-negligible oscillations. The unit segment is discretised by means of 50 elements, over which a unit velocity field is prescribed. The time step is chosen so that $dt = 0.1h/|v| = 0.002$. The discontinuity is initially placed at $x = 1/4$ and after 250 time steps, it is expected to have reached the position $x = 3/4$.

In Fig. 19 are shown the advected field for various values of the dimensionless coefficient $\gamma = \tau|\boldsymbol{v}|/h$. The Galerkin scheme ($\gamma = 0$) leads to strong oscillations, as already described by Donea and Huerta (2003). Using Eq. (A.2), one arrives to $\gamma = 0.045$, which leads to a desired removal of the oscillations through a small amount of numerical diffusion. Braun (2003) argues for a constant

[13] http://www.bccs.uni.no/

Thieulot, PEPI 188, 2011

We start from

$$\boldsymbol{M} \cdot \dot{\boldsymbol{T}} + \boldsymbol{K} \cdot \boldsymbol{T} = \boldsymbol{F}$$

Let us then write this equation at times t and $t + \delta t$:

$$\begin{aligned} \boldsymbol{M}(t) \cdot \dot{\boldsymbol{T}}(t) + \boldsymbol{K}(t) \cdot \boldsymbol{T}(t) &= \boldsymbol{F}(t) \\ \boldsymbol{M}(t+\delta t) \cdot \dot{\boldsymbol{T}}(t+\delta t) + \boldsymbol{K}(t+\delta t) \cdot \boldsymbol{T}(t+\delta t) &= \boldsymbol{F}(t+\delta t) \end{aligned}$$

To insure numerical stability, a second order accurate, mid-point implicit scheme ($\alpha = 0.5$) is used to represent the time derivative of temperature :

$$\frac{T(t+\delta t) - T(t)}{\delta t} = \alpha \dot{T}(t+\delta t) + (1-\alpha)\dot{T}(t)$$

- $\alpha = 0$: fully explicit scheme
- $\alpha = 1$: fully implicit scheme

One can multiply Eq.(1) by $1-\alpha$ and Eq. (1) by α and sum them :

$$
\begin{aligned}
(1-\alpha)\boldsymbol{M}(t)\cdot\dot{\boldsymbol{T}}(t) + (1-\alpha)\boldsymbol{K}(t)\cdot\boldsymbol{T}(t) &= (1-\alpha)\boldsymbol{F}(t) \\
+\alpha\boldsymbol{M}(t+\delta t)\cdot\dot{\boldsymbol{T}}(t+\delta t) + \alpha\boldsymbol{K}(t+\delta t)\cdot\boldsymbol{T}(t+\delta t) &\quad +\alpha\boldsymbol{F}(t+\delta t)
\end{aligned}
$$

Assuming $\boldsymbol{M}(t)\approx\boldsymbol{M}(t+\delta t)$, and $\boldsymbol{F}(t)\approx\boldsymbol{F}(t+\delta t)$, then

$$
\begin{aligned}
\boldsymbol{M}(t)\cdot\frac{T(t+\delta t)-T(t)}{\delta t} + (1-\alpha)\boldsymbol{K}(t)\cdot\boldsymbol{T}(t) &= \boldsymbol{F}(t) \\
+\alpha\boldsymbol{K}(t+\delta t)\cdot\boldsymbol{T}(t+\delta t) &
\end{aligned}
$$

and finally

$$
[\boldsymbol{M}(t)+\alpha\boldsymbol{K}(t)\,\delta t]\cdot\boldsymbol{T}(t+\delta t) = [\boldsymbol{M}(t)-(1-\alpha)\boldsymbol{K}(t)\,\delta t]\cdot\boldsymbol{T}(t) + \boldsymbol{F}(t)\,\delta t
$$

Another approach (to arrive at the same result):

$$\boldsymbol{M} \cdot \dot{\boldsymbol{T}} + \boldsymbol{K} \cdot \boldsymbol{T} = \boldsymbol{F}$$

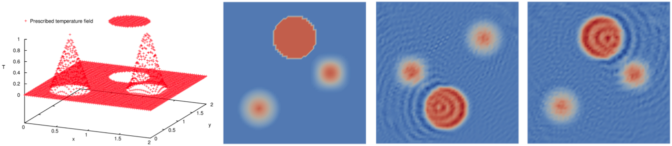
Looking at the Crank-Nicolson algorithm presented earlier, one can write:

$$\boldsymbol{M} \cdot \frac{\boldsymbol{T}(t+\delta t) - \boldsymbol{T}(t)}{\delta t} + \boldsymbol{K} \cdot (\alpha \boldsymbol{T}(t+\delta t) + (1-\alpha)\boldsymbol{T}(t)) = \boldsymbol{F}$$

which yields

$$\boxed{[\boldsymbol{M}(t) + \alpha \boldsymbol{K}(t)\ \delta t] \cdot \boldsymbol{T}(t+\delta t) = [\boldsymbol{M}(t) - (1-\alpha)\boldsymbol{K}(t)\ \delta t] \cdot \boldsymbol{T}(t) + \boldsymbol{F}(t)\ \delta t}$$

- Advection is notoriously difficult to get right:



- It often needs to be stabilised
- A standard approach is the Streamline Upwind Petrov Galerkin (SUPG) method.
- The advection matrix is computed as follows:

$$\boldsymbol{K}_a^e = \int_{x_k}^{x_{k+1}} (\boldsymbol{N}^\star)^T \rho c_p u \boldsymbol{B} dx \qquad \text{with} \qquad \boldsymbol{N}^\star = \boldsymbol{N} + \tau u \boldsymbol{B}$$

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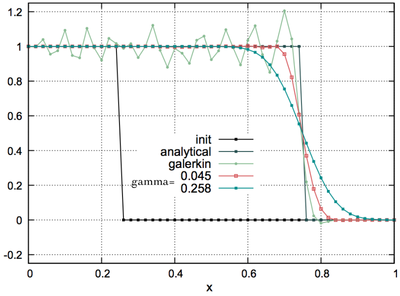
Thieulot, PEPI 188, 2011

Prove that in the SUPG case

$$(\boldsymbol{K}_a^e)_{SUPG} = \boldsymbol{K}_a^e + \rho c_p \frac{\tau u^2}{h_x} \left(\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right)$$

Reproduce the advection example presented in Appendix A of *Thieulot, PEPI 188, 2011* ($\rho = 1$, $c_p = 1$). Implement the SUPG stabilisation and vary the value of the τ parameter. Implement the implicit, explicit and mid-point algorithms look at their influence on the results.



where $\tau = \gamma h / v$.