Finite element models of steady state Rayleigh-Benard convection. 1) influence of the Rayleigh number

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1 Introduction

In parameterized convection models for planetary thermal evolution the heat transport characteristics of the convecting mantle are formulated in a pseudo steady state approximation. This is done by parameterization of the surface heatflux as a function of convective vigor, through the non-dimensional Nusselt number Nu. Nu is usually expressed in terms of the Rayleigh number Ra as $Nu \sim C\, Ra^{\beta}$, where C is a constant depending on the domain geometry (please do read section 1 of [15] for more information).

In this lab exercise you will investigate the characteristics of steady-state Rayleigh-Benard convection and determine the relation between the Nusselt and Rayleigh number experimentally, by means of numerical modelling. In particular you will measure the heatflow through the top surface of a 2D model of a convecting layer, as a function of the Rayleigh number, expressed in the temperature contrast across the convecting layer. This is done by a series of modelling experiments where the coupled equations for thermal convection are solved numerically using finite element methods.

The following sections contain descriptions of the numerical model and the experiments to be done.

2 The governing model equations

In this computerlab you will perform experiments with numerical solutions of the coupled equations describing thermal convection in an incompressible viscous fluid with infinite Prandtl number[1].

In what follows, the assumption is made that geological materials can be treated as fluids (with special properties) within the realm of continuum fluid mechanics. A Boussinesq approximation is applied, neglecting density variations in the equations except in the buoyancy term of the momentum conservation equation. We consider two-dimensional problems.

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \rho(T)\boldsymbol{g} &= \boldsymbol{0} && (1)\\
\boldsymbol{\nabla}\cdot\boldsymbol{v} &= 0 && (2)\\
\boldsymbol{\sigma} &= -p\boldsymbol{1} + \boldsymbol{s} && (3)\\
\boldsymbol{s} &= 2\mu\dot{\boldsymbol{\epsilon}} && (4)\\
\dot{\boldsymbol{\epsilon}} &= \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{v} + (\boldsymbol{\nabla}\boldsymbol{v})^T\right) && (5)\\
\rho_0 c_p \left(\frac{\partial T}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}T\right) &= \boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla}T) && (6)\\
\rho(T) &= \rho_0(1-\alpha T) && (7)
\end{aligned}
$$

Equation (1) is the momentum conservation equation and Eq. (2) is the mass conservation equation for incompressible fluids. One can resolve the stress tensor $\boldsymbol{\sigma}$ into its spherical part $-p\boldsymbol{1}$ and its stress deviation $\boldsymbol{s}$ (see Eq. (3)), where the deviatoric stress tensor is proportional to the strain rate tensor $\dot{\boldsymbol{\epsilon}}$ (see Eq.(4)) through the dynamic viscosity μ. Finally Eq. (5) relates the strain rate tensor to the velocity field.

Equations (1), (2), (3), (4) and (5) all together lead to the following form of the Stokes equations:

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot[\mu(\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\boldsymbol{v}^T)] - \boldsymbol{\nabla}p + \rho\boldsymbol{g} &= \boldsymbol{0} && (8)\\
\boldsymbol{\nabla}\cdot\boldsymbol{v} &= 0 && (9)
\end{aligned}
$$

Equation (11) is an elliptic equation characterized by the fact that changes in buoyancy and constitutive relationships anywhere in the domain have an immediate influence on the entire domain.

[1] In heat transfer problems, the Prandtl number controls the relative thickness of the momentum and thermal boundary layers. When Pr is small, it means that the heat diffuses quickly compared to the velocity (momentum).

symbol	meaning and dimension
$\boldsymbol{g}$	gravity acceleration vector ($m.s^{-2}$)
L_x, L_y	domain size (m)
p	pressure (Pa)
$\boldsymbol{s}$	deviatoric stress vector (Pa)
$\boldsymbol{v} = (u, v, w)$	velocity ($m.s^{-1}$)
$\dot{\boldsymbol{\epsilon}}$	strain-rate tensor (s^{-1})
λ	penalty coefficient ($Pa.s$)
μ	viscosity ($Pa.s$)
ρ, ρ_0	mass density ($kg.m^{-3}$)
$\boldsymbol{\sigma}$	stress tensor (Pa)
k	heat conductivity
c_p	heat capacity
α	thermal expansion

Table 1: Nomenclature

3 Numerical solution of the equations

3.1 The penalty method

In order to impose the incompressibility constraint, we shall use a widely used procedure, namely the the penalty method [2, 9]. It is implemented in the code you are going to use, which allows for the elimination of the pressure variable from the momentum equation (resulting in a reduction of the matrix size). Mathematical details on the origin and validity of the penalty approach applied to the Stokes problem can for instance be found in [4] or [8].

The penalty formulation of the mass conservation equation is based on a relaxation of the incompressibility constraint and writes

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} + \frac{p}{\lambda} = 0 \tag{10}$$

where λ is the penalty parameter, that can be interpreted (and has the same dimension) as a bulk viscosity. It is equivalent to say that the material is weakly compressible. It can be shown that if one chooses λ to be a relatively large number, the continuity equation $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ will be approximately satisfied in the finite element solution. The value of λ is often recommend to be 6 to 7 orders of magnitude larger than the shear viscosity [6, 9].

Eq. (10) can be used to eliminate the pressure in Eq. (11) so that the mass and momentum conservation equations fuse to become :

$$\boldsymbol{\nabla} \cdot [\mu(\boldsymbol{\nabla} \boldsymbol{v} + \boldsymbol{\nabla} \boldsymbol{v}^T)] + \lambda \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{v}) + \rho \boldsymbol{g} = \boldsymbol{0} \tag{11}$$

3.2 The finite element method

Introduced in the late 1950s, the finite element method (FEM) [9, 16] has emerged as one of the most powerful numerical methods so far devised.

Quadrilateral/hexahedral $Q_1 P_0$ elements (bi/tri-linear velocity, piecewise constant pressure) are used in this code. Despite the fact that they violate the Ladyzhenskaya, Babouska and Brezzi (LBB) stability condition [6], they remain a popular practical choice in mixed finite element approximation of incompressible materials.

This popularity can be explained by factors such as a) local mass conservation ; b) simple and uniform data structures and algebraic problems with manageable sizes and small bandwidths. The latter are of paramount importance for problems where geometry resolution requires very fine meshes and higher order elements can quickly lead to intractable algebraic problems in three space dimensions.

The physical domain Ω is broken up into elements, and a set of finite element basis functions is defined for each element so that functional representations of the independent variables can be constructed.

A thorough mathematical treatment of the finite element formulation of the equations gouverning the physics of the system is beyond the scope of this work, and has been exposed rigorously in some key references: the reader is referred to [6] or [8] for details on the theory and implementation of viscous incompressible flows, and to [12] for details on the theory and implementation of the heat transport equation. The FEM formulation of Eqs. (11) and (6) is presented succinctly in the next section.

Even though Eqs. (11) and (6) are coupled through the viscosity and density dependences on temperature and/or velocity, these equations are traditionally not solved in a coupled manner. The obtention of a new set of variables $(\boldsymbol{v}, p, T)$ at a given time is the product of a three-stage process:

1. solve for velocity field
2. recover pressure field from velocity field

3. solve for temperature

The size of the assembled FE matrix grows like the square of the total number of nodes, but this matrix is very sparse. Contrarily to optimised FE codes, no appropriate sparse storage scheme (lower triangular compressed sparse column, or CSC) is used in the code you are going to use. This strongly limits the number of elements/nodes which a grid can count.

The Galerkin finite element equation corresponding to Eq. (11) is

$$(\boldsymbol{K}_\mu + \boldsymbol{K}_\lambda) \cdot \boldsymbol{v} = \boldsymbol{B}$$

with

$$\boldsymbol{K}_\mu = \int_\Omega \boldsymbol{B}^T \cdot \boldsymbol{D}_\mu \cdot \boldsymbol{B} \, d\Omega$$

$$\boldsymbol{K}_\lambda = \int_\Omega \boldsymbol{B}^T \cdot \boldsymbol{D}_\lambda \cdot \boldsymbol{B} \, d\Omega$$

$$\boldsymbol{B} = \int_\Omega \boldsymbol{N}^T \rho \boldsymbol{g} \, d\Omega$$

$$\boldsymbol{D}_\mu^{2D} = \mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{D}_\lambda^{2D} = \lambda \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and where $\boldsymbol{N}$ is the vector of shape functions, and $\boldsymbol{B}$ is the matrix of spatial derivatives of the shape functions.

The finite element equation corresponding to the heat transfer equation is

$$\boldsymbol{M}_c \cdot \frac{\partial \boldsymbol{T}}{\partial t} + (\boldsymbol{K}_a + \boldsymbol{K}_d) \cdot \boldsymbol{T} = \boldsymbol{F}$$

where

$$\boldsymbol{M}_c = \int_\Omega \boldsymbol{N}^T \rho c_P \boldsymbol{N} \, d\Omega$$

$$\boldsymbol{K}_a = \int_\Omega (\boldsymbol{N}^\star)^T \rho c_P \boldsymbol{v} \cdot \boldsymbol{B} \, d\Omega \qquad (12)$$

$$\boldsymbol{K}_d = \int_\Omega \boldsymbol{B}^T k \boldsymbol{B} \, d\Omega$$

$$\boldsymbol{F} = \int_\Omega \boldsymbol{N}^T H \, d\Omega$$

where $\boldsymbol{T}$ is the vector of the nodal temperatures and $\boldsymbol{v}$ is the vector of nodal velocities.

4 Two-dimensional convection in a unit box

This benchmark deals with the 2-D thermal convection of a fluid of infinite Prandtl number in a rectangular closed cell. In what follows, we will focus on the case 1a, 1b, and 1c experiments as shown in [3]: steady convection with constant viscosity in a square box.

The temperature is fixed to zero on top and to ΔT at the bottom, with reflecting symmetry at the sidewalls (i.e. $\partial_x T = 0$) and there are no internal heat sources. Free-slip conditions are implemented on all boundaries.

The Rayleigh number is given by

$$Ra = \frac{\alpha g_y \Delta T h^3}{\kappa \nu} = \frac{\alpha g_y \Delta T h^3 \rho^2 c_p}{k \mu} \qquad (13)$$

In what follows, I use the following parameter values: $L_x = L_y = 1$,$\rho_0 = c_P = k = \mu = 1$, $T_0 = 0$, $\alpha = 10^{-4}$, $g = 10^4 Ra$.

The initial temperature field is given by

$$T(x, y) = (1 - y) - 0.01 \cos(\pi x / L_x) \sin(\pi y / L_y) \qquad (14)$$

The perturbation in the initial temperature fields leads to a perturbation of the density field and sets the fluid in motion. Depending on the initial Rayleigh number, the system ultimately reaches a steady state after some time.

The root mean square of the velocity field in the whole domain is defined as follows:

$$v_{rms} = \left(\frac{1}{V_\Omega} \int_\Omega |\boldsymbol{v}|^2 \, dV \right)^{1/2} \qquad (15)$$

		Blankenbach et al	Tackley [13]
$Ra = 10^4$	V_{rms}	42.864947 ± 0.000020	42.775
	Nu	4.884409 ± 0.000010	4.878
$Ra = 10^5$	V_{rms}	193.21454 ± 0.00010	193.11
	Nu	10.534095 ± 0.000010	10.531
$Ra = 10^6$	V_{rms}	833.98977 ± 0.00020	833.55
	Nu	21.972465 ± 0.000020	21.998

Table 2: Steady state Nusselt number Nu and V_{rms} measurements as reported in the literature.

The Nusselt number (i.e. the mean surface temperature gradient over mean bottom temperature) is computed as follows [3]:

$$Nu = L_y \frac{\int \frac{\partial T}{\partial y}(y = L_y)dx}{\int T(y = 0)dx} \tag{16}$$

Note that in our case the denominator is equal to 1 since $L_x = 1$ and the temperature at the bottom is prescribed to be 1.

Finally, the steady state root mean square velocity and Nusselt number measurements are indicated in Table 2 alongside those of [3] and [13]. (Note that this benchmark was also carried out and published in other publications [14, 1, 7, 5, 11] but since they did not provide a complete set of measurement values, they are not included in the table.)

5 How to

The code is to be downloaded there `http://cedricthieulot.net/mantledynamics.html` .

Move the `.tar` file to the location of your choice. Then untar the file as follows:

```
> tar -xvf code_lab1_mumps.tar
```

In the terminal, change the working directory to the folder that was just created (`code_lab1_mumps`). Then, copy the folder `/aw/mumps` into in your code folder as follows:

```
cp -r /aw/mumps .
```

In order to compile the code, you only have to type

```
> make
```

In order to run the code, you only have to type

```
> ./simplefem
```

Please open `simplefem.f90` with the text editor of your choice and look carefully at the code. Determine where the inputs are given, how and where the outputs are written.

Note that in between each run you can (should ?) erase all previously obtained data with

```
> ./clean
```

6 Experiments

- Determine the Nusselt number at steady state for a range of Rayleigh numbers, starting from a subcritical value. Collect the Rayleigh and Nusselt numbers in a 2-column ascii file and produce a plot of Nu against Ra using double logarithmic axes with gnuplot.
- Determine the logarithmic slope or powerlaw index β defined in the introduction.
- produce such a curve for various grid resolutions. How can you explain the differences in the results ? Produce a plot of Nu as a function of the grid spacing. Discuss.
- Look at how the v_{rms} values at steady state depend on Ra.
- For three contrasting Ra values, plot the temperature profiles (data to be found in `Tavrg.dat`) on a single plot and discuss the obtained figure.

- Set the number of points in the horizontal directions to 25. Choose $Ra = 10^5$ and progressively increase the number of points in the vertical direction. Report on the variation of the Nu number at steady state as a function of the vertical resolution.
- Estimate the value of the critical Rayleigh number from your Nusselt number plot and investigate the difference with the value found in Rayleigh's linear stability analysis for a layer of depth h and infinite horizontal extent, $Ra_C = (27/4)\pi^4$.
- The modelling program produces data files containing snapshots of the resulting numerical solution of the temperature and velocity fields in a suitable format (`.vtu`) for visualization with graphics program *paraview*. Produce colorplots with paraview of the temperature field, for three contrasting Rayleigh number cases, and discuss them.
- Explore the effect of the aspect ratio of the domain on Ra_c and the slope β.
- Change the initial temperature profile to something more random, repeat some of these experiments. What can you conclude ?

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